

# A $U_q(\widehat{gl}(2|2))_1$ -VERTEX MODEL: CREATION ALGEBRAS AND QUASI-PARTICLES I

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**ABSTRACT.** The infinite configuration space of an integrable vertex model based on  $U_q(\widehat{gl}(2|2))_1$  is studied at  $q = 0$ . Allowing four particular boundary conditions, the infinite configurations are mapped onto the semi-standard supertableaux of pairs of infinite border strips. By means of this map, a weight-preserving one-to-one correspondence between the infinite configurations and the normal forms of a pair of creation algebras is established for one boundary condition. A pair of type-II vertex operators associated with an infinite-dimensional  $U_q(gl(2|2))$ -module  $\hat{V}$  and its dual  $\hat{V}^*$  is introduced. Their existence is conjectured relying on a free boson realization. The realization allows to determine the commutation relation satisfied by two vertex operators related to the same  $U_q(gl(2|2))$ -module. Explicit expressions are provided for the relevant R-matrix elements. The formal  $q \rightarrow 0$  limit of these commutation relations leads to the defining relations of the creation algebras. Based on these findings it is conjectured that the type II vertex operators associated with  $\hat{V}$  and  $\hat{V}^*$  give rise to part of the eigenstates of the row-to-row transfer matrix of the model. A partial discussion of the R-matrix elements introduced on  $\hat{V} \otimes \hat{V}^*$  is given.

## 1. INTRODUCTION

This paper continues the study of the  $U_q(\widehat{gl}(2|2))$ -vertex model started in [1]. There an integrable vertex model based on the vector representation of  $U_q(gl(2|2))$  and its dual is investigated. In the limit  $q \rightarrow 0$ , the action of the corner transfer matrix Hamiltonian on the space of half-infinite configurations takes a trigonal form provided that the configurations obey a particular boundary condition. A one-to-one correspondence between the half-infinite configurations and the weight states of a reducible level-one module of  $U_q(\widehat{sl}(2|2))/\mathcal{H}$  with grade  $-n$  is observed for  $n \leq 3$ . Here the grade corresponds to the diagonal element of the corner transfer matrix Hamiltonian. The reducible module decomposes into one weakly integrable irreducible module and one nonintegrable irreducible module of  $U_q(\widehat{sl}(2|2))/\mathcal{H}$ . Both are highest weight modules.

Here the investigation of half-infinite configurations is supplemented taking into account a second boundary condition. Various choices of composite vertices and  $U_q(gl(2|2))$ -weights are considered. At  $q = 0$ , a one-to-one correspondence between the half-infinite configurations and the weight states of reducible or irreducible level-one modules of  $U_q(\widehat{sl}(2|2))/\mathcal{H}$  is found at the grades 0,  $-1$ ,  $-2$ . This correspondence is assumed to hold true at all grades. Similar as the modules relevant to the first boundary condition, the reducible modules decompose into a weakly integrable

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and a nonintegrable highest-weight module. These irreducible modules may be assembled in a reducible module in various ways. In each case, one possibility is distinguished by a simple free boson realization within the scheme employed in [1].

The arguments developed for the algebraic analysis of integrable  $U_q(\widehat{gl}(N))$ -vertex models ([2],[3],[4]) suggest that the full space of states is interpreted as a sum of tensor products each combining a highest weight module with a suitable dual (and therefore lowest weight) module. The levels of both constituents of the tensor product add to zero. According to the method presented in these references, the row-to-row transfer matrix should be described in terms of type I vertex operators associated with the vector representation  $W$  of  $U_q(gl(2|2))$  and its dual  $W^*$ . Type II vertex operators are expected to create the eigenstates of the row-to-row transfer matrix. In general, only a subset of all type II vertex operators existing for the affine quantum algebra at fixed level gives rise to such eigenstates. The correct choice of vertex operators follows from the decomposition of the tensor products modelling the space of states into irreducible components. For the  $U_q(\widehat{gl}(N))$ -vertex models this decomposition can be done at  $q = 0$  [3],[5] making use of the crystal base theory [6], [7]. The path space of the model is described in terms of a creation algebra whose generators can be seen as formal  $q \rightarrow 0$  limits of the appropriate type II vertex operators. This picture is based on the existence of the  $q \rightarrow 0$  limits of the creation operators acting on the true groundstate of the model. The existence of these limits has been conjectured and checked for the  $XXZ$ -model in [3].

In this study, a similar description is proposed for an infinite configuration space of the  $U_q(\widehat{gl}(2|2))$ -model at  $q = 0$ . The second boundary condition is imposed in both directions. In the limit  $q \rightarrow 0$ , the diagonal elements of the corner transfer matrix Hamiltonian decouple into a contribution depending only on the  $W$ -part of the half-infinite configuration and into a second part depending only on the  $W^*$ -part. For the boundary condition chosen here, the  $W$ -components and the  $W^*$ -components of the infinite configurations can be described separately in terms of two creation algebras  $\mathcal{A}$  and  $\mathcal{A}^*$ . According to their defining relations, bases of both algebras are given by their sets of normal forms. This has been proven in [5] for a quite general type of creation algebras including  $\mathcal{A}$  and  $\mathcal{A}^*$ . The infinite  $W$ -components ( $W^*$ -components) are in one-to-one correspondence with the normal forms of  $\mathcal{A}^*$  ( $\mathcal{A}$ ) supplemented by the unity.

To demonstrate these correspondences, two types of infinite border strips are employed. Border strips consisting of a finite number of rows or columns of finite length framed by either two infinite rows or two infinite columns are termed horizontal or vertical border strips, respectively. The semi-standard supertableaux of the horizontal (or vertical) border strips satisfying a suitable boundary condition are mapped onto the infinite  $W$ -(or  $W^*$ -)components and vice versa. Then a weight-preserving one-to-one map between the semi-standard supertableaux of the horizontal (vertical) border strips and the set of normal forms of  $\mathcal{A}^*$  ( $\mathcal{A}$ ) enlarged by the unity is given explicitly. A similar description can be constructed for the infinite configuration space restricted by the first boundary condition in both directions. If different boundary conditions are imposed in the left and right direction, the simple separation into the  $W$ -and  $W^*$ -components is lost. These cases will be considered in a separate publication.

The defining relations of  $\mathcal{A}$  and  $\mathcal{A}^*$  are expected to emerge from the formal  $q \rightarrow 0$  limits of the commutation relations satisfied by particular type II vertex

operators. Suitable vertex operators should be associated with a reducible, infinite-dimensional highest weight module  $\check{V}$  in the case of  $\mathcal{A}$  and with its dual module  $\check{V}^*$  in the case of  $\mathcal{A}^*$ . The module  $\check{V}$  decomposes into a one-dimensional module and an infinite-dimensional irreducible module. The latter coincides with the module  $V$  introduced in [1]. Based on a free boson realization for one component, the existence of such vertex operators is conjectured. The commutation relations of two vertex operators are governed by the R-matrices related to  $\check{V}$  and  $\check{V}^*$ . The free boson realizations allow to determine the normalisation of the R-matrices defined on  $\check{V} \otimes \check{V}$  or  $\check{V}^* \otimes \check{V}^*$ . Evaluation of their  $q \rightarrow 0$  limits formally leads to the defining relations of  $\mathcal{A}$  or  $\mathcal{A}^*$ . It seems more difficult however to demonstrate the compatibility of the commutation relations involving both types of vertex operators with the separation into  $W$ - and  $W^*$ -components found at  $q = 0$ . In the mixed case it is less obvious how to make use of  $q \rightarrow 0$  limits of single R-matrix elements. The  $q \rightarrow 0$  limits of the R-matrix elements on  $\check{V} \otimes \check{V}$  or  $\check{V}^* \otimes \check{V}^*$  reflect the structure of the irreducible components of these tensor products. Hence this structure underlies the defining relations of  $\mathcal{A}$  or  $\mathcal{A}^*$ . This motivates the search for an R-matrix action on the irreducible components of the tensor products  $\check{V}^* \otimes \check{V}$  or  $\check{V} \otimes \check{V}^*$ . This action is obtained on  $\check{V}^* \otimes \check{V}$  only for a partial range of the spectral parameter. Its formal  $q \rightarrow 0$  limit does not distinguish between different irreducible components.

From the above findings it may be conjectured that a part of the eigenstates of the row-to-row transfer matrix is generated by the type II vertex operators associated with  $\check{V}$  and  $\check{V}^*$ . A full discussion of the mixed R-matrix elements is more conveniently given in context with the R-matrix elements encountered in the case of mixed boundary conditions and is therefore relegated to a forthcoming publication.

The main findings presented here are formulated by conjecture 1 in section 3, conjectures 2 and 3 in section 6 and by the results 1-3 in section 5.

The paper is organised as follows. For the convenience of the reader, subsection 2.1 collects notations and recalls the free boson realization of  $U_q(\widehat{gl}(2|2))$  at level-one used in [1]. Subsection 2.2 gives a short account of the vertex model and the structure of the half-infinite configuration space subject to the boundary condition chosen in [1]. The second boundary condition is considered in section 3. Since the analysis is quite analogous to the one presented in [1], the outline is kept short here. In section 4, the infinite configuration space is mapped onto the semi-standard supertableaux. For one boundary condition, a one-to-one correspondence between the semi-standard supertableaux and the normal forms of two creation algebras is specified in section 5. Section 6 deals with the type II vertex operators related to the creation algebras. In section 7, the mixed case is investigated. Appendices A and C contain some details relegated from sections 5 and 7. Some properties of the R-matrix associated with the infinite-dimensional module and the list of explicit expressions for the R-matrix elements are given in appendix B.

## 2. THE MODEL

**2.1. The quantum affine superalgebra  $U_q(\widehat{gl}(2|2))$ .** The integrable vertex model investigated in [1] is based on the quantum affine superalgebra  $U_q(\widehat{gl}(2|2))$ . Defining relations in terms of the Chevalley or Drinfeld basis as well as references regarding

the representation theory of the algebra are given in [1].  $U_q(\widehat{gl}(2|2))$  is an associative  $\mathbb{Z}_2$ -graded algebra over  $\mathbb{C}[[q-1]]$  with generators  $E_n^{k,\pm}$ ,  $k = 1, 2, 3$ ,  $n \in \mathbb{Z}$  and  $\Psi_{\pm n}^{l,\pm}$ ,  $l = 1, 2, 3, 4$ ,  $n \in \mathbb{Z}_+$ , the central element  $c$  and the grading operator  $d$ . All simple roots are chosen odd. A  $\mathbb{Z}_2$  grading  $|\cdot|$  is defined by  $|E_n^{k,\pm}| = 1 \ \forall k, n$  and  $|q^c| = |d| = |\Psi_{\pm n}^{l,\pm}| = 0 \ \forall l, n$ . The defining relations of  $U_q(\widehat{gl}(2|2))$  in terms of the Drinfeld basis can be found in [1].  $U_q(\widehat{sl}(2|2))$  is the subalgebra with generators  $E_n^{k,\pm}$ ,  $n \in \mathbb{Z}$  and  $\Psi_{\pm n}^{k,\pm}$ ,  $n \in \mathbb{Z}_+$  with  $k = 1, 2, 3$ , the central element  $q^c$  and the grading operator  $d$ .  $U'_q(\widehat{gl}(2|2))$  and  $U'_q(\widehat{sl}(2|2))$  denote the superalgebras obtained by discarding the grading operator  $d$  from the set of generators of  $U_q(\widehat{gl}(2|2))$  or  $U_q(\widehat{sl}(2|2))$ , respectively.

On any  $U_q(\widehat{gl}(2|2))$ -module considered here,  $c$  acts as a scalar taking the value 0 or 1. This value is referred to as the level of the module. Parts of the subsequent analysis involve a free boson realisation of  $U_q(\widehat{gl}(2|2))$  applying to the level-one case. The generating functions

$$(1) \quad E_n^{k,\pm}(z) = \sum_{n \in \mathbb{Z}} E_n^{k,\pm} z^{-n-1} \quad \Psi_{\pm n}^{l,\pm}(z) = \sum_{n \geq 0} \Psi_{\pm n}^{l,\pm} z^{-n}$$

and the grading operator  $d$  can be expressed in terms of six sets  $\{\varphi^l, \varphi_0^l, \varphi_n^l, l = 1, 2, 3, 4; n \in \mathbb{Z}\}$  and  $\{\beta^{\bar{l}}, \beta_0^{\bar{l}}, \beta_n^{\bar{l}}, \bar{l} = 1, 2; n \in \mathbb{Z}\}$  of bosonic oscillators. These oscillators satisfy the commutation relations

$$(2) \quad \begin{aligned} [\varphi_n^l, \varphi_m^{l'}] &= \delta_{l,l'} \delta_{n+m,0} \frac{[n]^2}{n} \quad n, m \neq 0 \\ [\varphi_n^l, \varphi_0^{l'}] &= i \delta_{l,l'} \end{aligned}$$

and

$$(3) \quad \begin{aligned} [\beta_n^{\bar{l}}, \beta_m^{\bar{l}'}] &= -n \delta_{\bar{l},\bar{l}'} \delta_{n+m,0} \quad n, m \neq 0 \\ [\beta_n^{\bar{l}}, \beta_0^{\bar{l}'}] &= -i \delta_{\bar{l},\bar{l}'} \end{aligned}$$

where  $[n] \equiv \frac{q^n - q^{-n}}{q - q^{-1}}$ . The currents  $\Psi^{l,\pm}(z)$  are realized by

$$(4) \quad \Psi^{4,\pm}(z) = q^{\pm(\varphi_0^1 - i\varphi_0^4)} \exp\left(\pm(q - q^{-1}) \sum_{n>0} (\varphi_{\pm n}^1 - i\varphi_{\pm n}^4) z^{\mp n}\right)$$

and

$$(5) \quad \Psi^{l,\pm}(z) = q^{\mp i^l(\varphi_0^{l+1} + i\varphi_0^l)} \exp\left(\mp(q - q^{-1}) \sum_{n>0} i^l (\varphi_{\pm n}^{l+1} + i\varphi_{\pm n}^l) z^{\mp n}\right)$$

for  $l = 1, 2, 3$ . To express the remaining generating functions, it is convenient to introduce the deformed free fields

$$(6) \quad \varphi^{l,\pm}(z) = \varphi^l - i\varphi_0^l \ln z + i \sum_{n \neq 0} \frac{q^{\mp \frac{1}{2}|n|}}{[n]} \varphi_n^l z^{-n}$$

and

$$(7) \quad \beta^{\bar{l}}(z) = \beta^{\bar{l}} - i\beta_0^{\bar{l}} \ln z + i \sum_{n \neq 0} \frac{1}{n} \beta_n^{\bar{l}} z^{-n}$$

Then the generating functions  $E^{k\pm}(z)$  can be written

$$(8) \quad E^{k,\pm}(z) =: \exp\left(\mp i^{k+1}(\varphi^{k+1,\pm}(z) + i\varphi^{k,\pm}(z))\right) : \exp(\pm i\pi\delta_{k,1}\varphi_0^3) X^{k,\pm}(z)$$

with

$$(9) \quad \begin{aligned} -X^{1,-}(z) &= X^{2,+}(z) = \frac{1}{z(q-q^{-1})} \left( : \exp(\beta^1(q^{-1}z)) : - : \exp(\beta^1(qz)) : \right) \\ X^{1,+}(z) &= X^{2,-}(z) =: \exp(-\beta^1(z)) : \\ X^{3,+}(z) &= \frac{1}{z(q-q^{-1})} \left( : \exp(\beta^2(q^{-1}z)) : - : \exp(\beta^2(qz)) : \right) \\ X^{3,-}(z) &=: \exp(-\beta^2(z)) : \end{aligned}$$

The grading operator  $d$  is characterised by the properties

$$(10) \quad w^{-d} E^{k,\pm}(z) w^d = w E^{k,\pm}(wz) \quad w^{-d} \Psi^{l,\pm}(z) w^d = \Psi^{l,\pm}(wz)$$

With (4), (5) and (8), this implies

$$(11) \quad d = -\frac{1}{2} \sum_{l=1}^4 (\varphi_0^l)^2 - \sum_{l=1}^4 \sum_{n>0} \frac{n^2}{[n]^2} \varphi_{-n}^l \varphi_n^l + \frac{1}{2} \sum_{\bar{l}=1,2} \beta_0^{\bar{l}} (\beta_0^{\bar{l}} - i) + \sum_{\bar{l}=1,2} \sum_{n>0} \beta_{-n}^{\bar{l}} \beta_n^{\bar{l}}$$

Expressions (4), (5), (8) and (11) satisfy the defining relations of the Drinfeld basis of  $U_q(\widehat{gl}(2|2))$  with  $c$  replaced by the scalar 1. Later analysis of Fock spaces associated to the bosonic oscillators requires four further fields  $\eta^{\bar{l}}(z)$  and  $\xi^{\bar{l}}(z)$  introduced by

$$(12) \quad \eta^{\bar{l}}(z) = \sum_{n \in \mathbb{Z}} \eta_n^{\bar{l}} z^{-n-1} =: e^{-\beta^{\bar{l}}(z)} : \quad \xi^{\bar{l}}(z) = \sum_{n \in \mathbb{Z}} \xi_n^{\bar{l}} z^{-n} =: e^{\beta^{\bar{l}}(z)} :$$

with  $\bar{l} = 1, 2$ . The relations (3) imply  $\xi_n^{\bar{l}} \eta_m^{\bar{l}'} + \eta_m^{\bar{l}'} \xi_n^{\bar{l}} = \delta_{\bar{l}, \bar{l}'} \delta_{n+m, 0}$  and  $\xi_n^{\bar{l}} \xi_m^{\bar{l}'} + \xi_m^{\bar{l}'} \xi_n^{\bar{l}} = \eta_n^{\bar{l}} \eta_m^{\bar{l}'} + \eta_m^{\bar{l}'} \eta_n^{\bar{l}} = 0$ . Both  $\eta_0^{\bar{l}}$  and  $\eta_0^{\bar{l}'}$  commute with all generators (4), (5) and (8). In section 6, the free boson realisation of the superalgebra will be employed to establish the commutation relations of various type-II vertex operators.

For various purposes, it is convenient to write  $\Psi^{l,\pm}(z)$  in terms the generators  $h_l$  and  $H_n^l$  with  $l = 1, 2, 3, 4$  and  $n \in \mathbb{Z} \setminus 0$ :

$$(13) \quad \Psi^{l,\pm}(z) = q^{\pm h_l} \exp\left(\pm(q - q^{-1}) \sum_{n>0} H_{\pm n}^l z^{\mp n}\right) \quad l = 1, 2, 3, 4$$

Equations (4) and (5) yield

$$(14) \quad h_4 = \varphi_0^1 - i\varphi_0^4 \quad h_l = -i^l (\varphi_0^{l+1} + i\varphi_0^l)$$

$$(15) \quad H_n^4 = \varphi_n^1 - i\varphi_n^4 \quad H_n^l = -i^l (\varphi_n^{l+1} + i\varphi_n^l)$$

with  $l = 1, 2, 3, n \neq 0$ . The quantum superalgebras generated by  $E_0^{k,\pm}$  and  $q^{\pm h_l}$  with  $k, l = 1, 2, 3$  or  $k = 1, 2, 3, l = 1, 2, 3, 4$  are denoted by  $U_q(\widehat{sl}(2|2))$  or  $U_q(\widehat{gl}(2|2))$ , respectively. The generators  $h_1 + h_3$  and  $H_n^1 + H_n^3$  with  $n \neq 0$  constitute the commutative algebra  $\mathcal{H}$ . All generators of  $U_q'(\widehat{sl}(2|2))$  commute with  $\mathcal{H}$ .

In terms of the basis  $\{\tau_l\}_{1 \leq l \leq 4}$  with the bilinear form  $(\tau_l, \tau_{l'}) = (-1)^{l+1} \delta_{l,l'}$ , the classical simple roots  $\bar{\alpha}_l$  are written  $\bar{\alpha}_l = (-1)^{l+1} (\tau_l + \tau_{l+1})$  for  $l = 1, 2, 3$  and  $\bar{\alpha}_4 = \tau_1 - \tau_4$ . The classical weights  $\bar{\Lambda}_l$  with  $l = 1, 2, 3, 4$  are given by  $\bar{\Lambda}_l = \sum_{l'=1}^l \tau_{l'} - \frac{1}{2} \sum_{l'=1}^4 \tau_{l'}$ . In addition, an affine root  $\delta$  and the corresponding affine weight  $\Lambda_0$

with the properties  $(\Lambda_0, \Lambda_0) = (\delta, \delta) = (\tau_l, \Lambda_0) = (\tau_l, \delta) = 0$  and  $(\Lambda_0, \delta) = 1$  are introduced. Then the set of simple roots is expressed by  $\alpha_0 = \delta - \bar{\alpha}_1 - \bar{\alpha}_2 - \bar{\alpha}_3$  and  $\alpha_l = \bar{\alpha}_l$  for  $l = 1, 2, 3, 4$ . The weight lattice is the free Abelian group  $P = \oplus_{l=0}^4 \mathbb{Z}\Lambda_l + \mathbb{Z}\delta$  with  $\Lambda_l = \bar{\Lambda}_l + \Lambda_0$  for  $l = 1, 2, 3$  and  $\Lambda_4 = \bar{\Lambda}_4$ .  $P$  and its dual lattice  $P^* = \oplus_{l=0}^4 \mathbb{Z}h_l + \mathbb{Z}d$  with  $h_0 = c - h_1 - h_2 - h_3$  can be identified via the bilinear form  $(,)$  by setting  $\alpha_l = h_l$  and  $d = \Lambda_0$ .

**2.2. The vertex model.** The vertex model is constructed from the four-dimensional  $U'_q(\widehat{sl}(2|2))$ -module  $W$  with basis  $\{w_k\}_{0 \leq k \leq 3}$  and the dual module  $W^*$  with basis  $\{w_k^*\}_{0 \leq k \leq 3}$ . Their  $U'_q(\widehat{sl}(2|2))$ -structures are given by

$$(16) \quad \begin{aligned} h_j w_k &= (-1)^{j+1} (\delta_{j,k+1} + \delta_{j,k}) w_k \\ \Psi_{\pm n}^{j,\pm} w_k &= \pm (-1)^{j+1} (q - q^{-1}) q^{\pm n(2-\delta_{j,2})} (\delta_{j,k+1} + \delta_{j,k}) w_k \quad n > 0 \\ h_j w_k^* &= (-1)^j (\delta_{j,k+1} + \delta_{j,k}) w_k^* \\ \Psi_{\pm n}^{j,\pm} w_k^* &= \pm (-1)^j (q - q^{-1}) q^{\pm n\delta_{j,2}} (\delta_{j,k+1} + \delta_{j,k}) w_k^* \quad n > 0 \end{aligned}$$

with  $j = 1, 2, 3$  and

$$(17) \quad \begin{aligned} E_n^{j,+} w_j &= (-1)^{j+1} q^{n(2-\delta_{j,2})} w_{j-1} \\ E_n^{j,-} w_{j-1} &= q^{n(2-\delta_{j,2})} w_j \\ E_n^{j,+} w_{j-1}^* &= -q^{(n+2)\delta_{j,2}-1} w_j^* \\ E_n^{j,-} w_j^* &= (-1)^{j+1} q^{(n-2)\delta_{j,2}+1} w_{j-1}^* \end{aligned}$$

$\forall n$ . Extension of (17) and (16) to  $U'_q(\widehat{gl}(2|2))$ -structures is not unique. A convenient choice is

$$(18) \quad \begin{aligned} h_4 w_k &= (\delta_{k,0} - \delta_{k,3}) w_k \\ \Psi_{\pm n}^{4,\pm} w_k &= \pm (q - q^{-1}) (q^{\pm 3n} \delta_{k,0} - q^{\pm n} \delta_{k,3}) w_k \quad n > 0 \\ h_4 w_k^* &= -(\delta_{k,0} - \delta_{k,3}) w_k^* \\ \Psi_{\pm n}^{4,\pm} w_k^* &= \mp (q - q^{-1}) (q^{\mp n} \delta_{k,0} - q^{\pm n} \delta_{k,3}) w_k^* \quad n > 0 \end{aligned}$$

The modules  $W$  and  $W^*$  are attributed alternately to the horizontal as well as the vertical lines of the lattice composing the integrable vertex model. To each of the four types of elementary vertices, a spectral parameter  $z$  or  $(q^2 w)^{\pm 1} z$  is assigned as shown in figure 1. Boltzmann weights are assigned to the elementary vertices depending on the spectral parameter and on the configuration of basis elements  $\{w_k\}_{0 \leq k \leq 3}$  or  $\{w_k^*\}_{0 \leq k \leq 3}$  on the joining links. Their values follow from the R-matrices intertwining the action of  $U_q(\widehat{gl}(2|2))$  on the tensor products of two evaluation modules (71) (see [1]). Four neighbouring vertices may be viewed as a composite vertex of type A or B as illustrated in figure 1. In the limit  $q \rightarrow 0$ , the Boltzmann weights of a composite vertex provide a well-defined map  $(W \otimes W^*)^{\otimes 2} \rightarrow (W \otimes W^*)^{\otimes 2}$  (type A) or  $(W^* \otimes W)^{\otimes 2} \rightarrow (W^* \otimes W)^{\otimes 2}$  (type B). Due to the particular spectral inhomogeneity chosen, the maps are invertible. These properties allow to consider the Hamiltonian of a corner transfer matrix at  $q = 0$ . A northwest corner transfer matrix built from composite vertices of type A or type B acts on the half-infinite configurations  $(\dots \otimes v_{k_4} \otimes v_{k_3} \otimes v_{k_2} \otimes v_{k_1})$  subject to a suitable boundary condition. These configurations are specified by  $v_{2r} = w_{2r}$ ,  $v_{2r-1} = w_{2r-1}^*$  for type A and by  $v_{2r} = w_{2r}^*$ ,  $v_{2r-1} = w_{2r-1}$  for type B. The space of all configurations

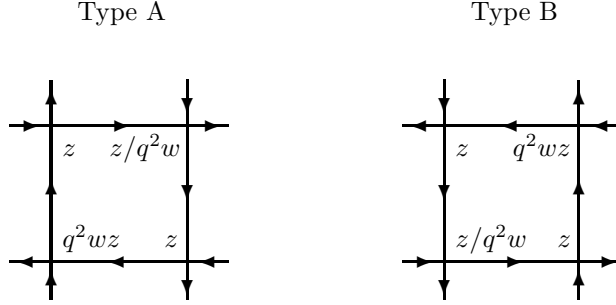


FIGURE 1. **Composite vertices and spectral parameters**  
 Arrows pointing right or upwards (left or downwards) indicate lattice links related to the module  $W$  ( $W^*$ ).

$(\dots \otimes v_{k_4} \otimes v_{k_3} \otimes v_{k_2} \otimes v_{k_1})$  with  $k_r = k$  for almost all  $r$  may be called  $\Omega_A^{(k)}$  in the type A case and  $\Omega_B^{(k)}$  in the type B case. In the following, the values  $k = 1$  and  $k = 3$  are considered. Analysis and structure of the results for  $\Omega_A^{(1)}$  and  $\Omega_B^{(1)}$  are similar to the procedure and findings described in [1] for the spaces  $\Omega_A^{(3)}$  and  $\Omega_B^{(3)}$ . The action of the CTM Hamiltonians becomes triangular in the limit  $q \rightarrow 0$ . Moreover, in this limit the diagonal elements decouple into a part depending only on the labels  $k_{2r}$  and another part determined by the entries  $k_{2r-1}$  only.

Denoting the diagonal element for the configuration  $(\dots \otimes w_{k_4} \otimes w_{k_3}^* \otimes w_{k_2} \otimes w_{k_1}^*) \in \Omega_A^{(k)}$  by  $h_{(\dots, k_4, k_3, k_2, k_1); (\dots, k_4, k_3, k_2, k_1)}$ , the two contributions are specified by

$$(19) \quad h_{(\dots, k_4, k_3, k_2, k_1); (\dots, k_4, k_3, k_2, k_1)} = - \sum_{r=1}^{\infty} r (x_{k_{2r+1}, k_{2r-1}} + y_{k_{2r+2}, k_{2r}})$$

with

$$(20) \quad x_{k_1, k_2} = y_{k_2, k_1} = \begin{cases} 0 & \text{if } k_1 > k_2 \text{ or } k_1 = k_2 = 1, 3; \\ 1 & \text{if } k_1 < k_2 \text{ or } k_1 = k_2 = 0, 2. \end{cases}$$

The diagonal element for the configuration  $(\dots \otimes w_{k_4}^* \otimes w_{k_3} \otimes w_{k_2}^* \otimes w_{k_1}) \in \Omega_B^{(k)}$  is given by

$$(21) \quad - \sum_{r=1}^{\infty} r (x_{k_{2r+2}, k_{2r}} + y_{k_{2r+1}, k_{2r-1}})$$

In view of the triangular action of the corner transfer matrix Hamiltonians,  $x_{k_1, k_2}$  and  $y_{k_1, k_2}$  may be called the quasi-energy functions of the vertex model.

A  $U_q(\widehat{gl}(2|2))$ -weight for a configuration  $(\dots \otimes w_{k_4} \otimes w_{k_3}^* \otimes w_{k_2} \otimes w_{k_1}^*) \in \Omega_A^{(k)}$  or  $(\dots \otimes w_{k_4}^* \otimes w_{k_3} \otimes w_{k_2}^* \otimes w_{k_1}) \in \Omega_B^{(k)}$  follows unambiguously from a  $U_q(\widehat{gl}(2|2))$ -reference weight  $(\bar{h}_1^{ref}, \bar{h}_2^{ref}, \bar{h}_3^{ref}, \bar{h}_4^{ref})$  introduced for the configuration  $(\dots \otimes w_k \otimes$

$w_k^* \otimes w_k \otimes w_k^*$  or  $(\dots \otimes w_k^* \otimes w_k \otimes w_k^* \otimes w_k)$ , respectively. For an arbitrary configuration in  $\Omega_A^{(k)}$ , the  $U_q(gl(2|2))$ -weight is given by

$$(22) \quad h_l(\dots \otimes w_{k_4} \otimes w_{k_3}^* \otimes w_{k_2} \otimes w_{k_1}^*) = \left\{ \bar{h}_l^{ref} + \bar{h}_l^{A,k}(\dots, k_4, k_3, k_2, k_1) \right\} (\dots \otimes w_{k_4} \otimes w_{k_3}^* \otimes w_{k_2} \otimes w_{k_1}^*)$$

with

$$(23) \quad \begin{aligned} \bar{h}_1^{A,k}(\dots, k_4, k_3, k_2, k_1) &= -\bar{h}_3^{A,k}(\dots, k_4, k_3, k_2, k_1) = \\ &= -\delta_{k,1} \sum_{r=1}^{\infty} (\delta_{k_{2r},2} + \delta_{k_{2r},3} - \delta_{k_{2r-1},2} - \delta_{k_{2r-1},3}) \\ &\quad + \delta_{k,3} \sum_{r=1}^{\infty} (\delta_{k_{2r},0} + \delta_{k_{2r},1} - \delta_{k_{2r-1},0} - \delta_{k_{2r-1},1}) \\ \bar{h}_2^{A,k}(\dots, k_4, k_3, k_2, k_1) &= \delta_{k,1} \sum_{r=1}^{\infty} (\delta_{k_{2r},0} + \delta_{k_{2r},3} - \delta_{k_{2r-1},0} - \delta_{k_{2r-1},3}) \\ &\quad - \delta_{k,3} \sum_{r=1}^{\infty} (\delta_{k_{2r},1} + \delta_{k_{2r},2} - \delta_{k_{2r-1},1} - \delta_{k_{2r-1},2}) \\ \bar{h}_4^{A,k}(\dots, k_4, k_3, k_2, k_1) &= \delta_{k,1} \sum_{r=1}^{\infty} (\delta_{k_{2r},0} - \delta_{k_{2r},3} - \delta_{k_{2r-1},0} + \delta_{k_{2r-1},3}) \\ &\quad + \delta_{k,3} \sum_{r=1}^{\infty} (2\delta_{k_{2r},0} + \delta_{k_{2r},1} + \delta_{k_{2r},2} - 2\delta_{k_{2r-1},0} - \delta_{k_{2r-1},1} - \delta_{k_{2r-1},2}) \end{aligned}$$

For any configuration  $(\dots \otimes w_{k_4}^* \otimes w_{k_3} \otimes w_{k_2}^* \otimes w_{k_1}) \in \Omega_B^{(k)}$ , the weight reads

$$(24) \quad h_l(\dots \otimes w_{k_4}^* \otimes w_{k_3} \otimes w_{k_2}^* \otimes w_{k_1}) = \left\{ \bar{h}_l^{ref} - \bar{h}_l^{A,k}(\dots, k_4, k_3, k_2, k_1) \right\} (\dots \otimes w_{k_4}^* \otimes w_{k_3} \otimes w_{k_2}^* \otimes w_{k_1})$$

A suitable choice of reference weights for the spaces  $\Omega_A^{(k)}$  and  $\Omega_B^{(k)}$  is

$$(25) \quad \begin{aligned} h_l^{ref} &= \delta_{l,2} + \delta_{l,4} & \text{for } k=1, \\ h_l^{ref} &= 0 \quad \forall l & \text{for } k=3. \end{aligned}$$

Another reference weight for  $\Omega_A^{(k)}$  is provided by adding the  $U_q(gl(2|2))$ -weight of  $w_k^*$  to the weight (25):

$$(26) \quad h_l^{ref} = \delta_{l,4} + 2\delta_{l,2} - \delta_{l,1} \quad \text{for } k=1,$$

$$(27) \quad h_l^{ref} = \delta_{l,4} - \delta_{l,3} \quad \text{for } k=3.$$

The weight assignments (22) with (25) and (26) will be called  $\bar{h}^{A,k}$  and  $\bar{h}'^{A,k}$ , respectively. Similarly, adding the  $U_q(gl(2|2))$ -weight of  $w_k$  to (25) gives a second reference weight for  $\Omega_B^{(k)}$ :

$$(28) \quad \begin{aligned} h_l^{ref} &= \delta_{l,1} + \delta_{l,4} & \text{for } k=1, \\ h_l^{ref} &= \delta_{l,3} - \delta_{l,4} & \text{for } k=3. \end{aligned}$$



The assignments (24) with (25) and (28) are denoted by  $\bar{h}^{B,k}$  and  $\bar{h}'^{B,k}$ . In the following section, a further reference weight

$$(29) \quad \begin{aligned} h_l^{ref} &= -s\delta_{l,1} + (1+s)\delta_{l,2} - (1+2s')\delta_{l,4} & \text{for } k=1, \\ h_l^{ref} &= s\delta_{l,3} + s'\delta_{l,4} & \text{for } k=3 \end{aligned}$$

with arbitrary  $s'$  and  $s \notin \mathbb{Z}$  will be taken into account both for  $\Omega_A^{(1)}$  and  $\Omega_B^{(1)}$ . The assignments (24) with (29) will be referred to as  $\bar{h}^{C,k}$ .

In [1], the half-infinite configurations of the spaces  $\Omega_A^{(3)}$  and  $\Omega_B^{(3)}$  are compared to the weight vectors of reducible level-one modules of  $U_q(\widehat{sl}(2|2))/\mathcal{H}$  denoted by  $\tilde{V}(\Lambda_0)$ ,  $\tilde{V}(\Lambda_1 + \Lambda_4)$  and  $\tilde{V}(2\Lambda_0 - \Lambda_3 + \Lambda_4)$ . The eigenvalues of the grading operator  $d$  and the generators  $h_l$ ,  $l = 1, 2, 3, 4$ , are called the grade and the  $U_q(\widehat{gl}(2|2))$ -weight of a weight vector. For a fixed weight assignment  $\bar{h}^A$ ,  $\bar{h}'^A$ ,  $\bar{h}^B$  or  $\bar{h}'^B$ , the  $U_q(\widehat{gl}(2|2))$ -weights of all configurations with the diagonal element of the CTM Hamiltonian given by  $-n$  are collected. At  $n = 0, 1, 2, 3$ , they are in one-to-one correspondence with the  $U_q(\widehat{gl}(2|2))$ -weights of all vectors with grade  $-n$  found in one of the reducible level-one modules. This correspondence may be assumed to hold true at any grade. Then the character of the reducible module can be expressed in terms of the quasi-energy functions. Due to the decomposition of the diagonal elements (19), (21), the character expressions factorise into two parts, each of them depending only on one quasi-energy function. Each of the reducible module can be decomposed into two irreducible level-one modules. One them is weakly integrable [8], the other nonintegrable. Table 1 specifies the relevant level-one modules for the four choices of composite vertices and assignment of weights. The weakly integrable irreducible module is listed left of the nonintegrable module.

space	weights	reducible module	irreducible modules	
$\Omega_A^{(3)}$	$\bar{h}^{A,3}$	$\tilde{V}(\Lambda_0)$	$V(\Lambda_0)$	$V(\Lambda_2 - \Lambda_4)$
$\Omega_A^{(3)}$	$\bar{h}'^{A,3}$	$\tilde{V}(2\Lambda_0 - \Lambda_3 - \Lambda_4)$	$V(2\Lambda_0 - \Lambda_3 - \Lambda_4)$	$V(-\Lambda_1 + 2\Lambda_2 - \Lambda_4)$
$\Omega_B^{(3)}$	$\bar{h}^{B,3}$	$\tilde{V}(\Lambda_0)$	$V(\Lambda_0)$	$V(\Lambda_2 - \Lambda_4)$
$\Omega_B^{(3)}$	$\bar{h}'^{B,3}$	$\tilde{V}(\Lambda_1 + \Lambda_4)$	$V(\Lambda_1 + \Lambda_4)$	$V(\Lambda_3 - \Lambda_4)$

TABLE 1. Assignment of  $U_q(\widehat{gl}(2|2))$ -weights and level-one modules for the configuration spaces  $\Omega_A^{(3)}$  and  $\Omega_B^{(3)}$

A similar analysis suggests that the weight states of the irreducible, nonintegrable level-one module  $V((1-s)\Lambda_0 + s\Lambda_3 + s'\Lambda_4)$ -module of  $U_q(\widehat{sl}(2|2))/\mathcal{H}$  correspond to the configurations if the reference weight  $\bar{h}^{C,3}$  is adopted.

### 3. THE SECOND BOUNDARY CONDITION

The configurations in  $\Omega_A^{(1)}$  and  $\Omega_B^{(1)}$  can be related to the weight vectors of level-one modules of  $U_q(\widehat{sl}(2|2))/\mathcal{H}$ . An assignment of weights  $\bar{h}^{A,1}$ ,  $\bar{h}'^{A,1}$ ,  $\bar{h}^{B,1}$ ,  $\bar{h}'^{B,1}$  or  $\bar{h}^{C,1}$  introduced in the previous section is fixed. Then the  $U_q(\widehat{gl}(2|2))$ -weights of all

configurations with a fixed value  $-n$  of the diagonal elements (19) or (21) are compared to the  $U_q(gl(2|2))$ -weights of all vectors with the associated grade present in a suitable level-one module. The value of the associated grade may equal the diagonal element or differ from it by a constant value. Consideration of the three lowest values  $n = 0, 1, 2$  indicates the appropriate modules. Three reducible level-one modules of  $U_q(\widehat{sl}(2|2))/\mathcal{H}$  denoted by  $\dot{V}(\Lambda_0)$ ,  $\dot{V}(2\Lambda_0 - \Lambda_3 + \Lambda_4)$  and  $\dot{V}(\Lambda_1 + \Lambda_4)$  account for the first four choices of composite vertices and  $U_q(gl(2|2))$ -weights. Similar as the reducible modules related to  $\Omega_A^{(3)}$  and  $\Omega_B^{(3)}$ , each of them decomposes into an irreducible weakly integrable and an irreducible nonintegrable module. In table 2, the level-one modules related to each case are listed. There the weakly integrable module appears left of the nonintegrable irreducible module. The irreducible, non-integrable level-one module  $V(-s\Lambda_1 + (s+1)\Lambda_2 - (1+2s')\Lambda_4)$  of  $U_q(\widehat{sl}(2|2))/\mathcal{H}$  accounts for the choice  $\bar{h}^{C,1}$ .

space	weights	reducible module	irreducible modules	
$\Omega_A^{(1)}$	$\bar{h}^{A,1}$	$\dot{V}(\Lambda_0)$	$V(\Lambda_0)$	$V(\Lambda_2 + \Lambda_4)$
$\Omega_A^{(1)}$	$\bar{h}'^{A,1}$	$\dot{V}(2\Lambda_0 - \Lambda_3 + \Lambda_4)$	$V(2\Lambda_0 - \Lambda_3 + \Lambda_4)$	$V(-\Lambda_1 + 2\Lambda_2 + \Lambda_4)$
$\Omega_B^{(1)}$	$\bar{h}^{B,1}$	$\dot{V}(\Lambda_0)$	$V(\Lambda_0)$	$V(\Lambda_2 + \Lambda_4)$
$\Omega_B^{(1)}$	$\bar{h}'^{B,1}$	$\dot{V}(\Lambda_1 + \Lambda_4)$	$V(\Lambda_1 + \Lambda_4)$	$V(\Lambda_3 + \Lambda_4)$

TABLE 2. Assignment of  $U_q(gl(2|2))$ -weights and level-one modules for the configuration spaces  $\Omega_A^{(1)}$  and  $\Omega_B^{(1)}$

A pair of irreducible modules can be assembled in a reducible module in various ways. In each case found in table 2, a particular reducible module is distinguished by a simple free boson realization in terms of the scheme given in section 2.2. These modules are obtained from the vectors

$$(30) \quad \kappa_0 = e^{\beta^1}|0\rangle \quad \kappa_1 = e^{i\varphi^1 + \beta^1}|0\rangle \quad \kappa_3 = \beta_{-1}^2 e^{-\varphi^4 + \beta^1 + \beta^2}|0\rangle$$

with the boson Fock vacuum  $|0\rangle$  characterised by the properties

$$(31) \quad \varphi_0^l|0\rangle = \beta_0^{\bar{l}}|0\rangle = 0 \quad \varphi_n^l|0\rangle = \beta_n^{\bar{l}}|0\rangle = 0 \quad \forall n > 0, l = 1, 2, 3, 4, \bar{l} = 1, 2$$

For the irreducible module  $V(-s\Lambda_1 + (1+s)\Lambda_2 - (1+2s')\Lambda_4)$ , the corresponding vector reads

$$(32) \quad \kappa_4 = e^{-s'i\varphi^1 - (s'-s)\varphi^2 + (1+s')(i\varphi^3 + \varphi^4) - s\beta^1}|0\rangle$$

The action of the grading operator on  $\kappa_I$  reads  $d\kappa_I = \frac{1}{2}(\delta_{I,0} - 1)\kappa_I$  for  $I = 0, 1, 3$  and  $d\kappa_4 = -\frac{1}{2}s(1+2s')\kappa_4$ . Arbitrary polynomials of the  $U_q(\widehat{gl}(2|2))$ -generators (4),(5) and (8) applied on  $\kappa_I$  give rise to vectors with the maximal value of their grades given by  $\frac{1}{2}(\delta_{I,0} + 2\delta_{I,3} - 1)$  for  $I = 0, 1, 3$  and by  $-\frac{1}{2}s(1+2s')$  for  $I = 4$ . If  $I = 0$ , the vectors with grade 0 form a reducible  $U_q(gl(2|2))$ -module which decomposes into the infinite-dimensional, irreducible module  $V(\Lambda_2 + \Lambda_4)$  and the one-dimensional module with weight  $(0, 0, 0, 0)$ . For  $I = 1$ , the vectors with grade  $-\frac{1}{2}$  furnish a reducible  $U_q(gl(2|2))$ -module which is decomposed into the four-dimensional module

$V(\bar{\Lambda}_1 + \bar{\Lambda}_4)$  and the infinite-dimensional module  $V(\bar{\Lambda}_3 + \bar{\Lambda}_4)$ . The latter are both irreducible. In the case  $I = 3$ , the vectors with grade  $\frac{1}{2}$  form the infinite-dimensional, irreducible  $U_q(\widehat{gl}(2|2))$ -module  $V(-\bar{\Lambda}_1 + 2\bar{\Lambda}_2 + \bar{\Lambda}_4)$ . Similarly, for  $I = 4$ , the vectors with grade  $-\frac{1}{2}s(1 + 2s')$  constitute the infinite-dimensional, irreducible  $U_q(\widehat{gl}(2|2))$ -module  $V(-s\bar{\Lambda}_1 + (1 + s)\bar{\Lambda}_2 - (1 + 2s')\bar{\Lambda}_4)$ . The  $U_q(\widehat{gl}(2|2))$ -modules obtained in this way can be viewed as the maximal-grade subspaces of level-one modules of  $U_q(\widehat{sl}(2|2))/\mathcal{H}$ . This suggests to examine the Fock spaces  $\mathring{\mathcal{F}}_I$  with  $I = 0, 1, 3, 4$  defined by

$$(33) \quad \mathring{\mathcal{F}}_I = \mathbb{C}[\varphi_{-1}^{\bar{l}_1}, \beta_{-1}^{\bar{l}_2}, \tilde{\varphi}_{-2}^{\bar{l}_3}, \beta_{-2}^{\bar{l}_4}, \dots] \\ \otimes \left( \bigoplus_{S_1, S_2, S_3 \in \mathbb{Z}} e^{S_1(\varphi^1 + \varphi^2 - \beta^1) + S_2(\varphi^2 - i\varphi^3 - \beta^1) + S_3(i\varphi^3 + \varphi^4 - \beta^2) + \beta^1 + \varpi_I} |0\rangle \right)$$

with  $\bar{l}_i = 1, 2$ ,

$$(34) \quad \begin{aligned} \tilde{\varphi}_{-n}^1 &\equiv \varphi_{-n}^1 - i\varphi_{-n}^2 + \varphi_{-n}^3 - i\varphi_{-n}^4 \\ \tilde{\varphi}_{-n}^2 &\equiv i\varphi_{-n}^2 + \varphi_{-n}^3 \end{aligned}$$

and

$$(35) \quad \begin{aligned} \varpi_0 &= 0 & \varpi_1 &= -\varphi^2 + \beta^1 & \varpi_3 &= \varphi^2 - \beta^1 \\ \varpi_4 &= s(\varphi^2 - \beta^1) - (1 + s')(i\varphi^1 + \varphi^2 - i\varphi^3 - \varphi^4) \end{aligned}$$

The remaining two linear combinations in  $\varphi_{-n}^l$  are not included in (33) since the Fock space  $\mathring{\mathcal{F}}_I$  should accommodate a module of  $U_q(\widehat{sl}(2|2))/\mathcal{H}$  rather than a  $U_q(\widehat{gl}(2|2))$ -module. Investigation of the next lower grades indicates that the required reducible modules are realized as restricted Fock spaces:

$$(36) \quad \begin{aligned} \mathring{V}(\Lambda_0) &= Ker_{\eta_0^2} \mathring{\mathcal{F}}_0 & \mathring{V}(\Lambda_1 + \Lambda_4) &= Ker_{\eta_0^2} \mathring{\mathcal{F}}_1 & \mathring{V}(2\Lambda_0 - \Lambda_3 + \Lambda_4) &= Ker_{\eta_0^2} \mathring{\mathcal{F}}_3 \\ \mathring{V}(-s\Lambda_1 + (1 + s)\Lambda_2 - (1 + 2s')\Lambda_4) &= Ker_{\eta_0^2} \mathring{\mathcal{F}}_4 \end{aligned}$$

The expression (11) for the grading operator  $d$  and the properties (31) yield

$$(37) \quad d e^{S_1(i\varphi^1 + \varphi^2 - \beta^1) + S_2(\varphi^2 - i\varphi^3 - \beta^1) + S_3(i\varphi^3 + \varphi^4 - \beta^2) + \beta^1 + \varpi_I} |0\rangle = \\ -\frac{1}{2} \left( S_1(S_1 - 1) + (S_2 - S_3)(S_2 - S_3 - 1) + \delta_{I,1} - \delta_{I,3} + s(1 + 2s')\delta_{I,4} \right) \cdot \\ \cdot e^{S_1(i\varphi^1 + \varphi^2 - \beta^1) + S_2(\varphi^2 - i\varphi^3 - \beta^1) + S_3(i\varphi^3 + \varphi^4 - \beta^2) + \beta^1 + \varpi_I} |0\rangle$$

for  $I = 0, 1, 3, 4$ . With (14) and (37), the  $U_q(\widehat{gl}(2|2))$ -weights of all vectors in  $Ker_{\eta_0^2} \mathring{\mathcal{F}}_I$  with grade  $-n$  are easily collected for small  $n$ . Then the one-to-one correspondence between the  $U_q(\widehat{gl}(2|2))$ -weights  $\bar{h}^{A,1}$ ,  $\bar{h}'^{A,1}$ ,  $\bar{h}^{B,1}$ ,  $\bar{h}'^{B,1}$  or  $\bar{h}^{C,1}$  of the configurations in  $\Omega_A^{(1)}$  or  $\Omega_B^{(1)}$  and the weights of the vectors in the modules (36) is readily verified for  $n = 0, 1, 2$ .

In the cases  $I = 0, 1, 3$ , the direct sum decomposition  $Ker_{\eta_0^2} \mathring{\mathcal{F}}_I = \xi_0^1 \eta_0^1 Ker_{\eta_0^2} \mathring{\mathcal{F}}_I \oplus \eta_0^1 \xi_0^1 Ker_{\eta_0^2} \mathring{\mathcal{F}}_I$  allows to separate the irreducible components of  $Ker_{\eta_0^2} \mathring{\mathcal{F}}_I$ . Boson

realizations for the irreducible  $U_q(\widehat{sl}(2|2))/\mathcal{H}$ -modules in table 2 are provided by

$$(38) \quad \begin{aligned} V(\Lambda_0) &= \eta_0^1 \text{Ker}_{\eta_0^2} \mathring{\mathcal{F}}_0 & V(\Lambda_2 + \Lambda_4) &= \text{Ker}_{\eta_0^1} \text{Ker}_{\eta_0^2} \mathring{\mathcal{F}}_0 \\ V(\Lambda_1 + \Lambda_4) &= \eta_0^1 \text{Ker}_{\eta_0^2} \mathring{\mathcal{F}}_1 & V(\Lambda_3 + \Lambda_4) &= \text{Ker}_{\eta_0^1} \text{Ker}_{\eta_0^2} \mathring{\mathcal{F}}_1 \\ V(2\Lambda_0 - \Lambda_3 + \Lambda_4) &= \eta_0^1 \text{Ker}_{\eta_0^2} \mathring{\mathcal{F}}_3 & V(-\Lambda_1 + 2\Lambda_2 + \Lambda_4) &= \text{Ker}_{\eta_0^1} \text{Ker}_{\eta_0^2} \mathring{\mathcal{F}}_3 \end{aligned}$$

Hence, the zero mode  $\eta_0^1$  annihilates the nonintegrable components. Some features of the reducible modules can be read from (30) and (38). Since  $\eta_0^1 \kappa_I \neq 0 \forall I$ , the vectors  $\kappa_I$  belong to the weakly integrable irreducible submodules. The coupling of both irreducible submodules is described by

$$(39) \quad \begin{aligned} E_0^{1,+} \kappa_I &\neq 0 & \eta_0^1 E_0^{1,+} \kappa_I &= E_0^{1,+} \eta_0^1 \kappa_I = 0 & I &= 0, 3 \\ E_0^{2,-} \kappa_J &\neq 0 & \eta_0^1 E_0^{2,-} \kappa_J &= E_0^{2,-} \eta_0^1 \kappa_J = 0 & J &= 0, 1 \end{aligned}$$

Thus  $E_0^{1,+} \kappa_0$ ,  $E_0^{2,-} \kappa_1$  and  $E_0^{1,+} \kappa_3$  are contained in the nonintegrable submodules. A vector with maximal grade in  $V(2\Lambda_0 - \Lambda_3 + \Lambda_4)$  is given by  $E_1^{2,-} E_0^{3,-} \kappa_3$ , for example. In the following, the notations  $\mathring{V}(\Lambda_0)$ ,  $\mathring{V}(\Lambda_1 + \Lambda_4)$  and  $\mathring{V}(2\Lambda_0 - \Lambda_3 + \Lambda_4)$  refer to the reducible modules realized by (36).

Provided that the correspondence between half-infinite configurations and weight states found for  $n = 0, 1, 2$  remains valid for all  $n \geq 0$ , the characters of the reducible modules can be written in terms of the energy functions (20). For the nonintegrable modules described above, well defined characters are introduced by

$$(40) \quad ch_{V(\cdot)}(\varrho, \rho_0, \rho_2, \rho_3) \equiv tr_{V(\cdot)} \varrho^d \rho_0^{\frac{1}{2}(h_1+h_2+h_3+h_4)} \rho_2^{-\frac{1}{2}(h_1+h_2-h_3-h_4)} \rho_3^{\frac{1}{2}(h_1+h_2+h_3-h_4)}$$

where  $|\rho_3| < 1$  and  $V(\cdot)$  denotes any of the modules listed in (36).

**Conjecture 1.** *The characters of the  $U_q(\widehat{sl}(2|2))/\mathcal{H}$ -modules  $\mathring{V}(\Lambda_0)$ ,  $\mathring{V}(\Lambda_1 + \Lambda_4)$ ,  $\mathring{V}(2\Lambda_0 - \Lambda_3 + \Lambda_4)$  and  $V(-s\Lambda_1 + (1+s)\Lambda_2 - (1+2s')\Lambda_4)$  are given by*

$$(41) \quad \begin{aligned} ch_{\mathring{V}(\Lambda_0)}(\varrho, \rho_0, \rho_2, \rho_3) &= \sum_{\{..., k_6, k_4, k_2\}} \varrho^{-\sum_{r>0} r y_{k_{2r+2}, k_{2r}}} \prod_{l=0,2,3} \rho_l^{\sum_{r>0} \delta_{k_{2r}, l}} \\ &\cdot \sum_{\{..., k_5, k_3, k_1\}} \varrho^{-\sum_{\bar{r}>0} \bar{r} x_{k_{2\bar{r}+1}, k_{2\bar{r}-1}}} \prod_{\bar{l}=0,2,3} \rho_{\bar{l}}^{-\sum_{r>0} \delta_{k_{2\bar{r}-1}, \bar{l}}} \end{aligned}$$

and

$$(42) \quad \begin{aligned} ch_{\mathring{V}(\Lambda_1 + \Lambda_4)}(\varrho, \rho_0, \rho_2, \rho_3) &= \varrho^{-\frac{1}{2}} ch_{\mathring{V}(\Lambda_0)}(\varrho, \rho_0, \rho_2, \rho_3) \\ ch_{\mathring{V}(2\Lambda_2 - \Lambda_3 + \Lambda_4)}(\varrho, \rho_0, \rho_2, \rho_3) &= \varrho^{\frac{1}{2}} ch_{\mathring{V}(\Lambda_0)}(\varrho, \rho_0, \rho_2, \rho_3) \\ ch_{\mathring{V}(-s\Lambda_1 + (1+s)\Lambda_2 - (1+2s')\Lambda_4)}(\varrho, \rho_0, \rho_2, \rho_3) &= \varrho^{-\frac{1}{2}s(1+2s')} ch_{\mathring{V}(\Lambda_0)}(\varrho, \rho_0, \rho_2, \rho_3) \end{aligned}$$

with  $|\rho_3| < 1$ . The sums in (41) are restricted by the requirement that  $k_r = 1$  for almost all  $r > 0$ .

In most of the remainder, the assignment  $\bar{h}^{A,1}$  is considered.

## 4. INFINITE BORDER STRIPS

Within the framework proposed in [2], [3], [4], the eigenstates of the row-to-row transfer matrix of the vertex model are created on the ground state by means of type II vertex operators. This general feature can be expected to apply to the  $U_q(\widehat{gl}(2|2))$ -model as well. Unfortunately, there is no simple method to single out the appropriate vertex operators among all those existing at the given level. In [3] and [5], the space of states of the  $XXZ$ -model and its higher spin generalisation has been decomposed at  $q = 0$ . The analysis involves a creation algebra whose defining relations can be viewed as formal  $q \rightarrow 0$  limits of the commutation relations satisfied by the appropriate type II vertex operators. A relation between the space of states at  $q = 0$  and the creation algebra is established by means of the domain wall picture and the crystal theory associated with the paths. The resulting expressions in terms of the generators of the creation algebra are interpreted as the  $q \rightarrow 0$ -limits of the  $n$ -particle eigenstates of the model. In the next section, two creation algebras relevant to the present model will be considered. Infinite border strips prove a useful tool for setting up the relation between their generators and the configuration space.

All following considerations refer to infinite configurations  $(\dots \otimes w_{j_2} \otimes w_{j_1}^* \otimes w_{j_0} \otimes w_{j_{-1}}^* \otimes w_{j_{-2}} \otimes w_{j_{-3}}^* \otimes \dots)$  subject to suitable boundary conditions. The set of all infinite configurations satisfying  $k_r = k \forall r > r_+ > 0$  and  $k_r = k' \forall r < r_- < 0$  with  $k, k' = 1, 3$  is denoted by  $\mathcal{K}_{k,k'}$ . Where convenient, the infinite components  $(\dots \otimes w_{j_2} \otimes w_{j_0} \otimes w_{j_{-2}} \otimes \dots)$  and  $(\dots \otimes w_{j_1}^* \otimes w_{j_{-1}}^* \otimes w_{j_{-3}}^* \otimes \dots)$  will be abbreviated by  $(\dots, j_2, j_0, j_{-2}, \dots)$  and  $(\dots, j_1, j_{-1}, j_{-3}, \dots)$ , respectively. All four choices of  $k, k'$  allow for a well-defined generalisation of the expression (19):

$$(43) \quad h_{(\dots, k_2, k_0, k_{-2}, k_{-4}, \dots); (\dots, k_2, k_0, k_{-2}, k_{-4}, \dots)} = - \sum_{r \in \mathbb{Z}} r y_{k_{2r+2}, k_{2r}}$$

$$(44) \quad h_{(\dots, k_3, k_1, k_{-1}, k_{-3}, \dots); (\dots, k_3, k_1, k_{-1}, k_{-3}, \dots)} = - \sum_{r \in \mathbb{Z}} r x_{k_{2r+1}, k_{2r-1}}$$

with  $x_{k_{2r+1}, k_{2r-1}}$  and  $y_{k_{2r+2}, k_{2r}}$  defined by (20).  $U_q(\widehat{gl}(2|2))$ -weights  $(\bar{h}_1, \bar{h}_2, \bar{h}_3, \bar{h}_4)$  compatible with the assignment  $\bar{h}^{A,1}$  defined by (22) and (25) are introduced for the infinite components of  $\mathcal{K}_{1,1}$  by

$$(45) \quad \begin{aligned} \bar{h}_1(\dots, k_2, k_0, k_{-2}, k_{-4}, \dots) &= -\bar{h}_3(\dots, k_2, k_0, k_{-2}, k_{-4}, \dots) \\ &= - \sum_{r \in \mathbb{Z}} (\delta_{k_{2r}, 2} + \delta_{k_{2r}, 3})(\dots, k_2, k_0, k_{-2}, k_{-4}, \dots) \\ \bar{h}_2(\dots, k_2, k_0, k_{-2}, k_{-4}, \dots) &= \sum_{r \in \mathbb{Z}} (\delta_{k_{2r}, 0} + \delta_{k_{2r}, 3})(\dots, k_2, k_0, k_{-2}, k_{-4}, \dots) \\ \bar{h}_4(\dots, k_2, k_0, k_{-2}, k_{-4}, \dots) &= \sum_{r \in \mathbb{Z}} (\delta_{k_{2r}, 0} - \delta_{k_{2r}, 3})(\dots, k_2, k_0, k_{-2}, k_{-4}, \dots) \end{aligned}$$

and

$$\begin{aligned}
(46) \quad \bar{h}_1(\dots, k_3, k_1, k_{-1}, k_{-3}, \dots) &= -\bar{h}_3(\dots, k_3, k_1, k_{-1}, k_{-3}, \dots) \\
&= \sum_{r \in \mathbb{Z}} (\delta_{k_{2r-1}, 2} + \delta_{k_{2r-1}, 3})(\dots, k_3, k_1, k_{-1}, k_{-3}, \dots) \\
\bar{h}_2(\dots, k_3, k_1, k_{-1}, k_{-3}, \dots) &= -\sum_{r \in \mathbb{Z}} (\delta_{k_{2r-1}, 0} + \delta_{k_{2r-1}, 3})(\dots, k_3, k_1, k_{-1}, k_{-3}, \dots) \\
\bar{h}_4(\dots, k_3, k_1, k_{-1}, k_{-3}, \dots) &= -\sum_{r \in \mathbb{Z}} (\delta_{k_{2r-1}, 0} - \delta_{k_{2r-1}, 3})(\dots, k_3, k_1, k_{-1}, k_{-3}, \dots)
\end{aligned}$$

For homogeneous vertex models related to quantum affine algebras, a one-to-one correspondence between the spin configurations and semi-standard super tableaux of skew Young diagrams has been demonstrated in [12]. A similar one-to-one correspondence between the half-infinite spin configurations of the present model and two types of semi-standard super tableaux of finite and half-infinite border strips is pointed out in [1]. Making use of this correspondence, pairs of infinite border strips can be related to the components  $(\dots \otimes w_{k_4} \otimes w_{k_2} \otimes w_{k_0} \otimes w_{k_{-2}} \otimes w_{k_{-4}} \otimes \dots)$  and  $(\dots \otimes w_{k_3}^* \otimes w_{k_1}^* \otimes w_{k_{-1}}^* \otimes w_{k_{-3}}^* \otimes w_{k_{-5}}^* \otimes \dots)$  of the infinite configurations  $(\dots \otimes w_{k_2} \otimes w_{k_1}^* \otimes w_{k_0} \otimes w_{k_{-1}}^* \otimes w_{k_{-2}} \otimes w_{k_{-3}}^* \otimes \dots)$ . These border strips consist of finitely many rows and columns of finite length assembled between either two half-infinite rows or two half-infinite columns. In the following, they will be referred to as horizontal or vertical border strips, respectively. As an example, figure 2 shows a horizontal border strip. The horizontal border strips are related to the components  $(\dots \otimes w_{k_4} \otimes w_{k_2} \otimes w_{k_0} \otimes w_{k_{-2}} \otimes w_{k_{-4}} \otimes \dots)$  and the vertical border strips to the components  $(\dots \otimes w_{k_3}^* \otimes w_{k_1}^* \otimes w_{k_{-1}}^* \otimes w_{k_{-3}}^* \otimes w_{k_{-5}}^* \otimes \dots)$ . A semi-standard super tableau of a horizontal or vertical border strip is obtained by assigning one of the numbers 0, 1, 2, 3 to each box such that the numbers of each two neighbouring boxes satisfy two rules:

- (1) If the side common to both boxes is vertical, then the number  $k_1$  in the left box and the number  $k_2$  in the right box fulfil

$$(47) \quad k_1 > k_2 \quad \text{or} \quad k_1 = k_2 = \begin{cases} 0, 2 & \text{vertical strip} \\ 1, 3 & \text{horizontal strip} \end{cases}$$

- (2) If the side common to both boxes is horizontal, then the number  $k_1$  in the upper box and the number  $k_2$  in the lower box fulfil

$$(48) \quad k_1 > k_2 \quad \text{or} \quad k_1 = k_2 = \begin{cases} 1, 3 & \text{vertical strip} \\ 0, 2 & \text{horizontal strip} \end{cases}$$

Almost all numbers attributed to the infinite strip are fixed by a boundary condition. The set of all semi standard super tableaux of horizontal (vertical) border strips with the number  $k$  given to almost all boxes of the lower half-infinite row (left half-infinite column) and the number  $k'$  given to almost all boxes in the upper half-infinite row (right half-infinite column) will be called  $\mathcal{B}_{k,k'}^h$  ( $\mathcal{B}_{k,k'}^v$ ). Each of the four choices  $k, k' = 1, 3$  is consistent with the rules (47), (48). The set  $\mathcal{B}_{k,k}^h$  ( $\mathcal{B}_{k,k}^v$ ) includes exactly one semi standard super tableau for the infinite border strip consisting of one single row (column). This tableau attributes the number  $k$  to each box. Excluding this tableau from  $\mathcal{B}_{k,k}^h$  ( $\mathcal{B}_{k,k}^v$ ) yields a set called  $\mathcal{B}_{k,k}^{h \setminus 0}$  ( $\mathcal{B}_{k,k}^{v \setminus 0}$ ).

A horizontal border strip with at least one finite row is characterised by the set  $R, (p_1, p_2, \dots, p_R; \bar{p}_1, \bar{p}_2, \dots, \bar{p}_{R-1})$  with  $R > 1$  and  $p_i, \bar{p}_i \in \mathbb{N}$ . This border strip contains  $R$  finite columns with more than one box. A vertical border strip with at least one finite column is described by the set  $R, (p_1, p_2, \dots, p_{R-1}; \bar{p}_1, \bar{p}_2, \dots, \bar{p}_R)$  with  $R > 1$  and  $p_i, \bar{p}_i \in \mathbb{N}$ . In both cases, the parameter  $p_i + 1$  ( $\bar{p}_i + 1$ ) specifies the number of the boxes contained in the  $i$ -th finite column (finite row) composed of at least two boxes. Here the counting proceeds from the right to the left end of a horizontal border strip or from the upper to the lower end of a vertical border strip. In the remainder, the counting of rows or columns refers only to rows or columns consisting of more than one box. The set  $R = 1, (p_1; \emptyset)$  with  $p_1 \in \mathbb{N}$  specifies a horizontal border strip composed of the lower and upper half-infinite row and a column containing  $p_1 + 1 \geq 2$  boxes. Similarly, the set  $R = 1, (\emptyset; \bar{p}_1)$  characterises a vertical border strip built from the left and right half-infinite columns and a row with  $\bar{p}_1 + 1 \geq 2$  boxes. Finally, the value  $R = 0$  refers to the border strips consisting of one single infinite row or column.

A horizontal (vertical) border strip with  $R \geq 1$  has exactly one semi-standard super tableau in  $\mathcal{B}_{1,1}^h$  ( $\mathcal{B}_{1,1}^v$ ) involving only the numbers 0 and 1. In a horizontal border strip, the  $p_i$  lower boxes of each column with  $p_i + 1 \geq 2$  boxes receive the number 0. All other boxes obtain the number 1. In case of a vertical border strip, the rightmost  $\bar{p}_i$  boxes in a row with  $\bar{p}_i + 1 \geq 2$  boxes receive the value 0. The number 1 is given to all other boxes. These tableaux will be called the reference labellings in the following. An example is illustrated by figure 2.

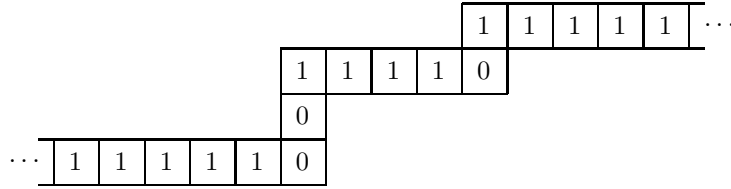


FIGURE 2. The reference labelling for the horizontal border strip with  $R = 2$  and parameters  $p_1 = 1, p_2 = 2, \bar{p}_1 = 4$ .

In order to define a mapping from a semi standard super tableau onto a component of an infinite configuration, a counting of the boxes needs to be specified. The leftmost box of the upper half-infinite row of a horizontal border strip or the lowest box in the right half-infinite column of a vertical border strip is counted as the  $r_0$ -th box for some  $r_0 \in \mathbb{Z}$ . The  $(r + 1)$ -th box is left of or below the  $r$ -th box for any  $r \in \mathbb{Z}$ . Then for a horizontal border strip with parameters  $R = 0, R = 1, (p_1; \emptyset)$  or  $R > 1, (p_1, \dots, p_R; \bar{p}_1, \dots, \bar{p}_{R-1})$  an arbitrary semi-standard super tableau in  $\mathcal{B}_{k,k'}^h$  may be considered. Given a fixed value of  $r_0$ , it is convenient to introduce numbers  $s_i, 1 \leq i \leq R$ , by

$$(49) \quad \begin{aligned} s_1 &= r_0 \\ s_i &= r_0 + \sum_{i_{<}=1}^{i-1} (p_{i_{<}} + \bar{p}_{i_{<}}) \quad 2 \leq i \leq R \end{aligned}$$

and

$$(50) \quad d^h(p_1, \dots, p_R; \bar{p}_1, \dots, \bar{p}_{R-1}; r_0) = \sum_{i=1}^R p_i \left( s_i + \frac{1}{2}(p_i - 1) \right)$$

A component  $(\dots \otimes w_{k_4} \otimes w_{k_2} \otimes w_{k_0} \otimes w_{k_{-2}} \otimes w_{k_{-4}} \otimes \dots)$  is associated with the semi-standard super tableau by identifying the number attributed to the  $(r+1)$ -th box with  $k_{-2r}$ . With (20) and (43) it is easily verified that the contribution of this component to the diagonal element of the CTM Hamiltonian coincides with (50).

For a vertical border strip with parameters  $R' = 0$ ,  $R' = 1, (\emptyset; \bar{p}'_1)$  or  $R' > 1, (p'_1, \dots, p'_{R'-1}; \bar{p}'_1, \dots, \bar{p}'_{R'})$  and the box counting fixed by the number  $r'_0$ , the number

$$(51) \quad d^v(p'_1, \dots, p'_{R'-1}; \bar{p}'_1, \dots, \bar{p}'_{R'}; r_0) = \sum_{i=1}^{R'} \bar{p}'_i \left( s'_i + \frac{1}{2}(\bar{p}'_i - 1) \right)$$

is introduced. Here  $s'_i$  is defined by (49) with  $r_0$ ,  $p_{i<}$ ,  $\bar{p}_{i<}$  replaced by  $r'_0$ ,  $p'_{i<}$  and  $\bar{p}'_{i<}$ , respectively. A semi standard super tableau in  $\mathcal{B}_{k,k'}^v$  is related to a component  $(\dots \otimes w_{k_3}^* \otimes w_{k_1}^* \otimes w_{k_{-1}}^* \otimes w_{k_{-3}}^* \otimes w_{k_{-5}}^* \otimes \dots)$  by identifying the number attributed to the  $(r+1)$ -th box with  $k_{-(2r-1)}$ . According to (20) and (44), the contribution of this component to the diagonal element of the CTM Hamiltonian equals (51).

Analogous statements apply to the components  $(\dots \otimes w_1 \otimes w_1 \otimes w_1 \otimes \dots)$  and  $(\dots \otimes w_1^* \otimes w_1^* \otimes w_1^* \otimes \dots)$  provided that

$$(52) \quad d^h(0, \emptyset) = d^v(\emptyset, 0) = 0$$

This is readily verified comparing the definitions (20) for  $y_{k_{2r+2}, k_{2r}}$  and  $x_{k_{2r+1}, k_{2r-1}}$  with the rules (47), (48). A  $U_q(gl(2|2))$ -weight is assigned to each semi standard super tableau in  $\mathcal{B}_{1,1}^h$  or  $\mathcal{B}_{1,1}^v$  via (46) or (45) and the above identifications.

The above prescription gives a one-to-one correspondence between the components  $(\dots \otimes w_{k_4} \otimes w_{k_2} \otimes w_{k_0} \otimes w_{k_{-2}} \otimes w_{k_{-4}} \otimes \dots)$  and  $(\dots \otimes w_{k_3}^* \otimes w_{k_1}^* \otimes w_{k_{-1}}^* \otimes w_{k_{-3}}^* \otimes w_{k_{-5}}^* \otimes \dots)$  of the configurations in  $\mathcal{K}_{k,k'}$  and the sets  $B_{k,k'}^h$  and  $B_{k,k'}^v$  of semi-standard super tableaux associated with infinite border strips. This correspondence will be used in the next section to describe the configuration space in terms of two creation algebras.

## 5. THE CREATION ALGEBRAS

**Definition 1.** *The creation algebra  $\mathcal{A}^*$  is generated by  $\{\phi_{j,t;m}^* | 0 \leq j \leq 3, t \in \mathbb{N}_0, m \in \mathbb{Z}\}$  over  $\mathcal{Z}$  subject to the defining relations*

$$(53) \quad \phi_{j_2, t_2; m_2}^* \phi_{j_1, t_1; m_1}^* = -\phi_{j_2, t_2; m_1+t_2+\theta_{j_2, j_1}}^* \phi_{j_1, t_1; m_2-t_2-\theta_{j_2, j_1}}^*$$

with

$$(54) \quad \theta_{j_2, j_1} = \begin{cases} 0, & \text{if } j_2 = 2; \\ 0, & \text{if } j_2 = 0, 3 \text{ and } j_1 = 0, 1; \\ 1, & \text{otherwise.} \end{cases}$$

A special case of the defining relations (53) is

$$(55) \quad \phi_{j_2, t_2; m+t_2+\theta_{j_2, j_1}}^* \phi_{j_1, t_1; m}^* = 0 \quad \forall t_1 \in \mathbb{N}_0, m \in \mathbb{Z}$$



For the subsequent analysis, it is useful to introduce the notion of normal forms [5]. The product

$$(56) \quad \phi_{j_n, t_n; m_n}^* \cdots \phi_{j_2, t_2; m_2}^* \phi_{j_1, t_1; m_1}^*$$

is called a normal form (of  $\mathcal{A}^*$ ) iff

$$(57) \quad m_{i+1} > m_i + t_{i+1} + \theta_{j_{i+1}, j_i} \quad \text{for } 1 \leq i < n$$

The set

$$(58) \quad B^* = \cup_{n \in \mathbb{N}} \{ \phi_{j_n, t_n; m_n}^* \cdots \phi_{j_1, t_1; m_1}^* | j_i = 0, 1, 2, 3, t_i \in \mathbb{N}_0, m_i \in \mathbb{Z} \text{ satisfy (57)} \}$$

provides a  $\mathbb{Z}$ -linear base of the algebra  $\mathcal{A}^*$ . This statement is a special case of corollary 2 proven in [5]. A  $U_q(\widehat{gl}(2|2))$ -weight  $(h_1^{\phi^*}, h_2^{\phi^*}, h_3^{\phi^*}, h_4^{\phi^*})$  is introduced for the generators  $\phi_{j, t; m}^*$  via

$$(59) \quad \begin{aligned} [h_l, \phi_{j, t; m}^*] &= h_l^{\phi^*}(j, t) \phi_{j, t; m}^*, \quad l = 1, 2, 3, 4 \\ h_1^{\phi^*}(j, t) &= -h_3^{\phi^*}(j, t) = -(t + 1 - \delta_{j, 2}) \\ h_2^{\phi^*}(j, t) &= t + 1 - \delta_{j, 0} \\ h_4^{\phi^*}(j, t) &= -(t - 1 + \delta_{j, 0} + 2\delta_{j, 1}) \end{aligned}$$

The commutator with the grading operator  $d$  is defined by

$$(60) \quad [d, \phi_{j, t; m}^*] = m \phi_{j, t; m}^*$$

Taking into account (59), (60) and (46), the set  $B^*$  of normal forms may be compared to the set  $\mathcal{B}_{1,1}^{h \setminus 0}$  of semi standard super tableaux.

For fixed values of  $r_0 \in \mathbb{Z}$  and  $p_1 \in \mathbb{N}$ , the set of all normal forms

$$(61) \quad \phi_{j_{p_1}, t_{p_1}; r_0 + p_1 - 1}^* \cdots \phi_{j_2, t_2; r_0 + 1}^* \phi_{j_1, t_1; r_0}^*$$

is called  $B^*(p_1; \emptyset; r_0)$ .

For  $r_0 \in \mathbb{Z}$  and  $(p_1, p_2, \dots, p_R; \bar{p}_1, \bar{p}_2, \dots, \bar{p}_{R-1})$  with  $p_i, \bar{p}_i \in \mathbb{N}$  and  $R > 1$ , the set of all normal forms  $\phi_{j_N, t_N; m_N}^* \cdots \phi_{j_2, t_2; m_2}^* \phi_{j_1, t_1; m_1}^*$  with

$$(62) \quad \begin{aligned} N &= \sum_{i=1}^R p_i, \quad m_1 = r_0, \\ m_{i+1} - m_i &= \begin{cases} \bar{p}_{\hat{R}} + 1 & \text{if } i = p_1 + p_2 + \dots + p_{\hat{R}} \text{ for } 1 \leq \hat{R} < R, \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

is denoted by  $B^*(p_1, p_2, \dots, p_R; \bar{p}_1, \bar{p}_2, \dots, \bar{p}_{R-1}; r_0)$ .

The set of all semi-standard super tableaux associated with the horizontal border strip  $R > 1, (p_1, \dots, p_R; \bar{p}_1, \dots, \bar{p}_{R-1})$  or  $R = 1, (p_1, \emptyset)$  with the box counting fixed by  $r_0$  can be mapped onto  $B^*(p_1, \dots, p_R; \bar{p}_1, \dots, \bar{p}_{R-1}; r_0)$  or  $B^*(p_1; \emptyset; r_0)$ , respectively. In particular, the reference labelling introduced in the previous section is mapped onto the normal form with  $j_r = 2, t_r = 0 \forall r$ . This amounts to attributing the generator  $\phi_{2, 0; r}^*$  to the  $r+1$ -th box if the reference labelling assigns the number 0 to this box. According to (46) and the correspondence between configurations and semi-standard super tableaux, the reference labelling is the only tableau of the border strip with the  $U_q(\widehat{gl}(2|2))$ -weight given by  $(0, N, 0, N)$ . Equation (59) specifies the only normal form in  $B^*(p_1, \dots, p_R; \bar{p}_1, \dots, \bar{p}_{R-1}; r_0)$  or  $B^*(p_1; \emptyset; r_0)$  with the same values.

An arbitrary semi-standard super tableau in  $\mathcal{B}_{1,1}^{h \setminus 0}$  with  $R \geq 1$  is mapped onto a normal form (62) in three steps. It is convenient to rewrite the normal form (62) as

$$\begin{aligned}
 (63) \quad & \phi_{j_{p_1}, t_{p_1}; s_1+p_1-1}^* \cdots \phi_{j_2, t_2; s_1+1}^* \phi_{j_1, t_1; s_1}^* \\
 & \cdot \phi_{j_{p_1+p_2}, t_{p_1+p_2}; s_2+p_2-1}^* \cdots \phi_{j_{p_1+2}, t_{p_1+2}; s_2+1}^* \phi_{j_{p_1+1}, t_{p_1+1}; s_2}^* \\
 & \cdot \phi_{j_{p_1+p_2+p_3}, t_{p_1+p_2+p_3}; s_3+p_3-1}^* \cdots \phi_{j_{p_1+p_2+2}, t_{p_1+p_2+2}; s_3+1}^* \phi_{j_{p_1+p_2+1}, t_{p_1+p_2+1}; s_3}^* \\
 & \cdots \\
 & \cdot \phi_{j_N, t_N; s_R+p_R-1}^* \cdots \phi_{j_{p_1+\dots+p_{R-1}+2}, t_{p_1+\dots+p_{R-1}+2}; s_R+1}^* \phi_{j_{p_1+\dots+p_{R-1}+1}, t_{p_1+\dots+p_{R-1}+1}; s_R}^*
 \end{aligned}$$

with the numbers  $s_i$  defined in (49). The factors in the  $i$ -th line in (63) are related to the numbers given to the boxes in the  $i$ -th column and the right neighbouring row as follows.

- (1) For  $1 \leq i \leq R$  and  $p_i \geq 2$ , the factors

$$\begin{aligned}
 & \phi_{j_{p_1+\dots+p_i}, t_{p_1+\dots+p_i}; s_i+p_i-1}^* \cdots \phi_{j_{p_1+\dots+p_{i-1}+3}, t_{p_1+\dots+p_{i-1}+3}; s_i+2}^* \\
 & \cdot \phi_{j_{p_1+\dots+p_{i-1}+2}, t_{p_1+\dots+p_{i-1}+2}; s_i+1}^*
 \end{aligned}$$

are determined according to the numbers attributed to the  $p_i$  lower boxes of the  $i$ -th column. Figure 3 specifies these factors for all cases allowed by the rules (47) and (48) for the tableaux of horizontal strips.

- (2) To obtain the factor  $\phi_{j_{p_1+\dots+p_{i-1}+1}, t_{p_1+\dots+p_{i-1}+1}; s_i}^*$  with  $i > 1$  and  $p_i \geq 1$ , the  $i$ -th column and the  $(i-1)$ -th finite row with length  $\bar{p}_{i-1}+1$  is taken into account. Figure 4 collects all possible cases together with the associated factors.
- (3) Depending on the numbers given to the boxes of the upper infinite column, the rightmost factor  $\phi_{j_1, t_1; s_1}^*$  is determined. All cases are listed in figure 5.

For a semi-standard super tableau of the border strip  $(p_1, \emptyset)$  with  $p_1 > 1$ , the normal form (61) is determined by steps (2) and (3). In the case  $(1; \emptyset)$ , only step (3) is required. A box on the lhs of figures 3-5 is drawn boldly if it is the  $(r+1)$ th box in the complete border strip and the corresponding expression on the rhs contains a generator  $\phi_{j, t; r}^*$ .

Each correspondence listed in figures 3-5 matches the  $U_q(gl(2|2))$ -weights introduced in (46) and (59). Moreover, the sum  $m_1 + m_2 + \dots + m_N$  coincides with the number  $d^h(p_1, \dots, p_R; \bar{p}_1, \dots, \bar{p}_{R-1}; r_0)$  introduced in (50). As a consequence of the rules (47)-(48), an arbitrary tableau in  $\mathcal{B}_{1,1}^{h \setminus 0}$  assigns the number 1 to all boxes of the lower half-infinite row except the rightmost of them. Thus the prescriptions given in figures 3-5 specify one normal form of  $B^*(p_1, \dots, p_R; \bar{p}_1, \dots, \bar{p}_{R-1}; r_0)$  (or  $B^*(p_1; \emptyset; r_0)$ ) for each tableau of the border strip with the parameters  $(p_1, \dots, p_R; \bar{p}_1, \dots, \bar{p}_{R-1})$  (or  $(p_1; \emptyset)$ ) and  $r_0$ . Due to the definition (57), the same prescriptions attribute exactly one semi-standard super tableau related for the parameters  $(p_1, \dots, p_R; \bar{p}_1, \dots, \bar{p}_{R-1})$  (or  $(p_1; \emptyset)$ ) and  $r_0$  to each normal form in  $B^*(p_1, \dots, p_R; \bar{p}_1, \dots, \bar{p}_{R-1}; r_0)$  (or  $B^*(p_1; \emptyset; r_0)$ ).

**Result 1.** *The semi-standard super tableaux in  $\mathcal{B}_{1,1}^{h \setminus 0}$  related a horizontal border strip with the parameters  $r_0, (p_1, \dots, p_R; \bar{p}_1, \dots, \bar{p}_{R-1})$  (or  $(p_1; \emptyset)$ ), and the normal*

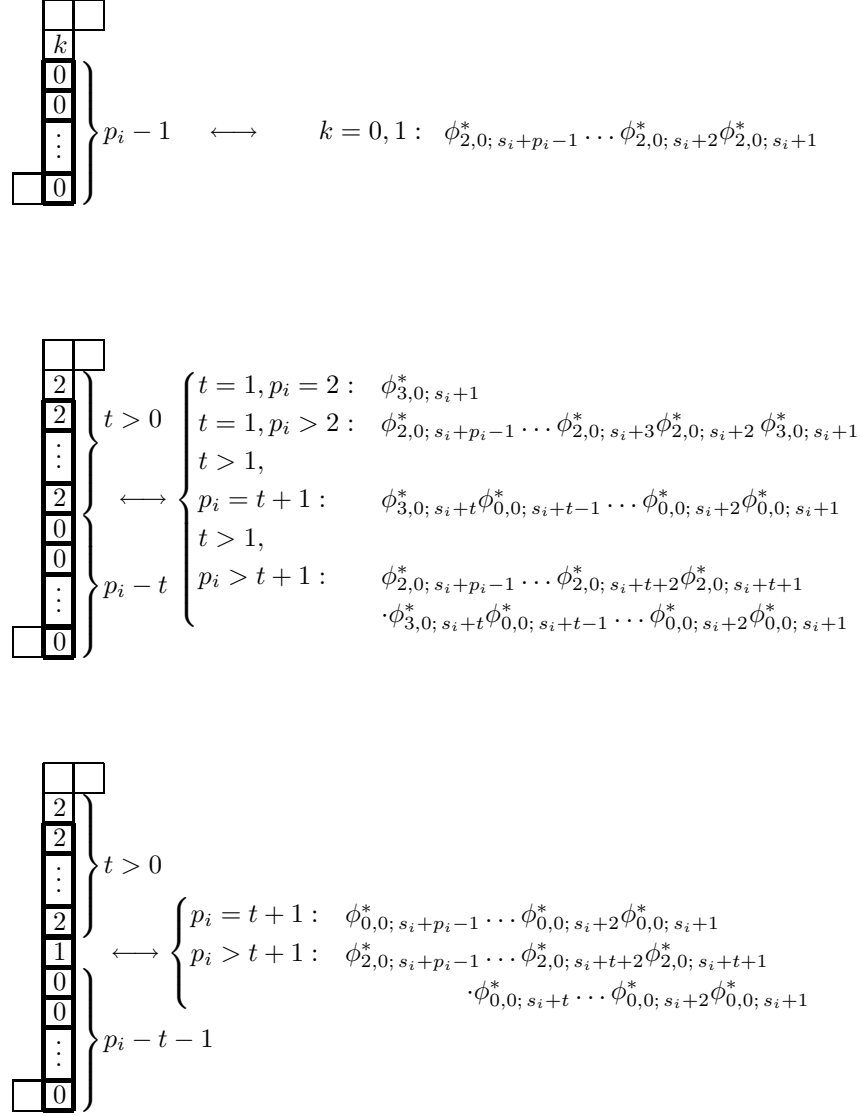


FIGURE 3. Mapping of the columns

forms in  $B^*(p_1, \dots, p_R; \bar{p}_1, \dots, \bar{p}_{R-1}; r_0)$  (or  $B^*(p_1; \emptyset; r_0)$ ) are in one-to-one correspondence. Hence, the tableaux in  $\mathcal{B}_{1,1}^{h \setminus 0}$  and the normal forms in  $B^*$  are in one-to-one correspondence.

The set  $\mathcal{B}_{1,1}^{v \setminus 0}$  of semi-standard super tableaux associated with the vertical border strips is related to the set of normal forms of the creation algebra  $\mathcal{A}$ .

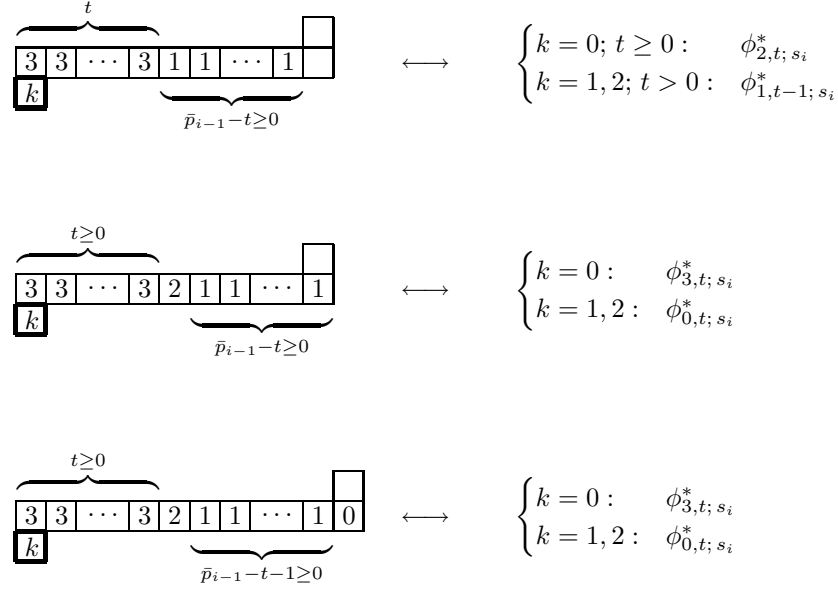


FIGURE 4. Mapping of the finite rows

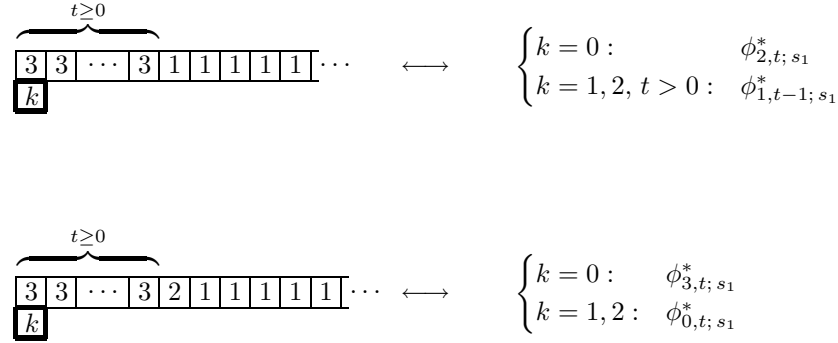


FIGURE 5. Mapping of the upper half-infinite row

**Definition 2.** The creation algebra  $\mathcal{A}$  is generated by  $\{\phi_{j,t;m} | 0 \leq j \leq 3, t \in \mathbb{N}_0, m \in \mathbb{Z}\}$  over  $\mathcal{Z}$  subject to the defining relations

$$(64) \quad \phi_{j_2,t_2;m_2} \phi_{j_1,t_1;m_1} = -\phi_{j_2,t_2;m_1+t_1+\theta_{j_1,j_2}} \phi_{j_1,t_1;m_2-t_1-\theta_{j_1,j_2}}$$

with  $\theta_{j,j'}$  defined by (54).

The procedure applying to the case of vertical border strips is quite analogous to the one outlined above for the horizontal border strips. A product

$$(65) \quad \phi_{j_n, t_n; m_n} \cdots \phi_{j_2, t_2; m_2} \phi_{j_1, t_1; m_1}$$

is referred to as a normal form (of  $\mathcal{A}$ ) iff

$$(66) \quad m_{i+1} > m_i + t_i + \theta_{j_i, j_{i+1}} \quad \text{for } 1 \leq i < n$$

According to corollary 2 in [5], the set

$$(67) \quad B = \cup_{n \in \mathbb{N}} \{ \phi_{j_n, t_n; m_n} \cdots \phi_{j_1, t_1; m_1} | j_i = 0, 1, 2, 3, t_i \in \mathbb{N}_0, m_i \in \mathbb{Z} \text{ satisfy (66)} \}$$

provides a  $\mathbb{Z}$ -linear basis of  $\mathcal{A}$ . A  $U_q(\widehat{gl}(2|2))$ -weight is defined for the generators  $\phi_{j, t; m}$  by

$$(68) \quad [h_l, \phi_{j, t; m}] = -h_l^{\phi^*}(j, t) \phi_{j, t; m}, \quad l = 1, 2, 3, 4$$

with  $h_l^{\phi^*}$  given by (59). A mapping of the set  $\mathcal{B}_{1,1}^{v \setminus 0}$  of semi-standard super tableaux onto the set  $B$  similar to the mapping of  $\mathcal{B}_{1,1}^{h \setminus 0}$  on  $B^*$  is specified in Appendix A. The set of all normal forms  $\phi_{j_{\bar{p}_1}, t_{\bar{p}_1}; r_0 + \bar{p}_1 - 1} \cdots \phi_{j_1, t_2; r_0 + 1} \phi_{j_1, t_1; r_0}$  with fixed  $r_0 \in \mathbb{Z}$  and  $\bar{p}_1 \in \mathbb{N}$  is denoted by  $B(\emptyset; \bar{p}_1)$ . For  $r_0 \in \mathbb{Z}$  and  $(p_1, p_2, \dots, p_R; \bar{p}_1, \bar{p}_2, \dots, \bar{p}_{R-1})$  with  $p_i, \bar{p}_i \in \mathbb{N}$  and  $R > 1$ , the set of all normal forms  $\phi_{j_{\bar{N}}, t_{\bar{N}}; m_{\bar{N}}} \cdots \phi_{j_2, t_2; m_2} \phi_{j_1, t_1; m_1}$  with

$$(69) \quad \begin{aligned} \bar{N} &= \sum_{i=1}^R \bar{p}_i, \quad m_1 = r_0, \\ m_{i+1} - m_i &= \begin{cases} p_{\hat{R}} + 1 & \text{if } i = \bar{p}_1 + \bar{p}_2 + \dots + \bar{p}_{\hat{R}} \text{ for } 1 \leq \hat{R} < R, \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

is denoted by  $B(p_1, p_2, \dots, p_{R-1}; \bar{p}_1, \bar{p}_2, \dots, \bar{p}_R; r_0)$ .

**Result 2.** *The semi-standard super tableaux in  $\mathcal{B}_{1,1}^{v \setminus 0}$  related to a vertical border strip with parameters  $r_0, (p_1, \dots, p_{R-1}; \bar{p}_1, \dots, \bar{p}_R)$  (or  $(\emptyset, \bar{p}_1)$ ) and the normal forms in  $B(p_1, \dots, p_{R-1}; \bar{p}_1, \dots, \bar{p}_R; r_0)$  (or  $B(\emptyset, \bar{p}_1)$ ) are in one-to-one correspondence.*

For a normal form  $\phi_{j_{\bar{N}}, t_{\bar{N}}; m_{\bar{N}}} \cdots \phi_{j_1, t_1; m_1}$  in  $B(p_1, \dots, p_{R-1}; \bar{p}_1, \dots, \bar{p}_R; r_0)$  or  $B(\emptyset, \bar{p}_1)$ , the sum  $m_1 + \dots + m_{\bar{N}}$  equals the number  $d^v(p_1, \dots, p_{R-1}; \bar{p}_1, \dots, \bar{p}_R; r_0)$  introduced in (51).

In the following section, the definition of the algebra  $\mathcal{A}^*$  (or  $\mathcal{A}$ ) is motivated by a naive limit of the commutation relations of type II vertex operators. Besides the components related to the generators  $\phi_{j, t; m}^*$  (or  $\phi_{j, t; m}$ ), the vertex operators have a further component equal to the unit. Each of the sets  $B^*, B$ , may be supplemented by the unit. The resulting sets are denoted by  $B_1^*$  and  $B_1$ , respectively. Formally, the unit may be assigned to the tableau in  $\mathcal{B}_{1,1}^h$  or  $\mathcal{B}_{1,1}^v$  attributing the number one to each box of the border strip given by the single infinite or half-infinite row or column. Application of the map between the components and the semi-standard supertableaux specified in section 4 yields a relation between the enlarged sets and the infinite configurations.

**Result 3.** *The infinite components  $(\dots \otimes w_{k_4} \otimes w_{k_2} \otimes w_{k_0} \otimes w_{k_{-2}} \otimes w_{k_{-4}} \otimes \dots)$  with  $k_r = 1$  for almost all  $r$  are in one-to-one correspondence with the set  $B_1^*$ . The infinite components  $(\dots \otimes w_{k_3}^* \otimes w_{k_1}^* \otimes w_{k_{-1}}^* \otimes w_{k_{-3}}^* \otimes w_{k_{-5}}^* \otimes \dots)$  with  $k_r = 1$  for almost all  $r$  are in one-to-one correspondence with the set  $B_1$ .*

## 6. TYPE II VERTEX OPERATORS

The type II vertex operators considered below are intertwiners of  $U$ -modules of the form

$$(70) \quad \Phi_{V_I}^{V V_J}(z) : V_I \longrightarrow V_z \otimes V_J$$

where  $V_I$  and  $V_J$  are level-one  $U_q(\widehat{sl}(2|2))/\mathcal{H}$ -modules.  $V$  with basis  $\{v_j\}$  denotes a  $U_q(\widehat{sl}(2|2))$ -module obtained from an infinite-dimensional  $U_q(gl(2|2))$ -module by means of the evaluation homomorphism [9], [10], [11]. The evaluation module  $V_z = V \otimes \mathbb{C}[z, z^{-1}]$  is endowed with a  $U_q(\widehat{sl}(2|2))$ -structure via

$$(71) \quad \begin{aligned} E_n^{k,\pm}(v_j \otimes z^m) &= E_n^{k,\pm} v_j \otimes z^{m+n} & \forall n \\ \Psi_{\pm n}^{l,\pm}(v_j \otimes z^m) &= \Psi_{\pm n}^{l,\pm} v_j \otimes z^{m \pm n} & n \geq 0 \\ d(v_j \otimes z^m) &= m v_j \otimes z^m \\ c(v_j \otimes z^m) &= 0 \end{aligned}$$

The vertex operator (70) is introduced as the formal series

$$(72) \quad \begin{aligned} \Phi_{V_I}^{V V_J}(z) &= \sum_j v_j \otimes \left( \Phi_{V_I}^{V V_J} \right)_j(z) \\ \left( \Phi_{V_I}^{V V_J} \right)_j(z) &= \sum_{m \in \mathbb{Z}} \left( \Phi_{V_I}^{V V_J} \right)_{j;m} z^{-m} \end{aligned}$$

In terms of the maps  $\left( \Phi_{V_I}^{V V_J} \right)_{j;m} : V_I \longrightarrow V_J$  the intertwining property reads

$$(73) \quad \begin{aligned} \Delta(a) \left\{ \sum_j \sum_{m \in \mathbb{Z}} (v_j \otimes z^{-m}) \otimes \left( \Phi_{V_I}^{V V_J} \right)_{j;m} \right\} \\ = \sum_j \sum_{m \in \mathbb{Z}} (-1)^{|a| \cdot (|v_j| + |\Phi_j|)} (v_j \otimes z^{-m}) \otimes \left( \Phi_{V_I}^{V V_J} \right)_{j;m} a \end{aligned}$$

$\forall a \in U_q(\widehat{gl}(2|2))$ . Here  $|v_j|$  and  $|\Phi_j|$  denote the  $\mathbb{Z}_2$ -gradings of  $v_j$  and the component  $\left( \Phi_{V_I}^{V V_J} \right)_{j;m}$ . In the remainder, their relation is fixed by  $|\Phi_j| = |v_j|$ . For the generators of  $U_q(gl(2|2))$ ,  $q^{\pm \frac{1}{2}c}$  and the grading operator  $d$ , the coproduct  $\Delta$  is defined by

$$(74) \quad \begin{aligned} \Delta(E_0^{k,+}) &= E_0^{k,+} \otimes 1 + q^{h_k} \otimes E_0^{k,+} & \Delta(E_0^{k,-}) &= E_0^{k,-} \otimes q^{-h_k} + 1 \otimes E_0^{k,-} \\ \Delta(q^{\pm h_l}) &= q^{\pm h_l} \otimes q^{\pm h_l} & k &= 1, 2, 3, l = 1, 2, 3, 4 \end{aligned}$$

and

$$(75) \quad \Delta(q^{\pm \frac{1}{2}c}) = q^{\pm \frac{1}{2}c} \otimes q^{\pm \frac{1}{2}c} \quad \Delta(d) = d \otimes 1 + 1 \otimes d$$

A partial information on the coproduct of the remaining Drinfeld generators of  $U_q(\widehat{gl}(2|2))$  is provided by the formulae [13]

$$\begin{aligned}
 (76) \quad \Delta(E_n^{k,+}) &= E_n^{k,+} \otimes q^{nc} + q^{2nc+h_k} \otimes E_n^{k,+} \\
 &\quad + \sum_{n'=0}^{n-1} q^{\frac{1}{2}(n+3n')c} \Psi_{n-n'}^{k,+} \otimes q^{(n-n')c} E_{n'}^{k,+} \quad \text{mod } N_- \otimes N_+^2 \\
 \Delta(E_{-n}^{k,+}) &= E_{-n}^{k,+} \otimes q^{-nc} + q^{-h_k} \otimes E_{-n}^{k,+} \\
 &\quad + \sum_{n'=1}^{n-1} q^{\frac{1}{2}(n-n')c} \Psi_{-n+n'}^{k,-} \otimes q^{-(n-n')c} E_{-n'}^{k,+} \quad \text{mod } N_- \otimes N_+^2 \\
 \Delta(E_n^{k,-}) &= E_n^{k,-} \otimes q^{h_k} + q^{nc} \otimes E_n^{k,-} \\
 &\quad + \sum_{n'=1}^{n-1} q^{(n-n')c} E_{n'}^{k,-} \otimes q^{-\frac{1}{2}(n-n')c} \Psi_{n-n'}^{k,+} \quad \text{mod } N_-^2 \otimes N_+ \\
 \Delta(E_{-n}^{k,-}) &= E_{-n}^{k,-} \otimes q^{-2mc-h_k} + q^{-nc} \otimes E_{-m}^{k,-} \\
 &\quad + \sum_{n'=0}^{n-1} q^{-(n-n')c} E_{-n'}^{k,-} \otimes q^{-\frac{1}{2}(n+3n')c} \Psi_{-n+n'}^{k,-} \quad \text{mod } N_-^2 \otimes N_+ \\
 \Delta(H_n^l) &= H_n^l \otimes q^{\frac{1}{2}nc} + q^{\frac{3}{2}nc} \otimes H_n^l \quad \text{mod } N_- \otimes N_+ \\
 \Delta(H_{-n}^l) &= H_{-n}^l \otimes q^{-\frac{3}{2}nc} + q^{-\frac{1}{2}nc} \otimes H_{-n}^l \quad \text{mod } N_- \otimes N_+
 \end{aligned}$$

for  $k = 1, 2, 3$ ,  $l = 1, 2, 3, 4$ , and  $n > 0$ .  $N_{\pm}$  and  $N_{\pm}^2$  are left  $\mathbb{Q}(q)[q^{\pm c}, q^{\pm h_l}, \Psi_{\pm n}^{l,\pm}]$ -modules generated by  $\{E_m^{k,\pm}\}_{k=1,2,3, m \in \mathbb{Z}}$  and  $\{E_m^{k,\pm} E_{m'}^{k',\pm}\}_{k,k'=1,2,3, m,m' \in \mathbb{Z}}$ , respectively. Equations (76) are readily verified using the Hopf algebra structure of  $U_q(\widehat{gl}(2|2))$  given in [14]. In context with vertex models based on the quantum affine algebra  $U_q(\widehat{gl}(N))$ , type II vertex operators associated with finite-dimensional evaluation modules of the algebra are considered. Expressions analogous to (76) allow to derive free boson expressions for a particular component of a vertex operator (see [2], [15], [16] and [17], for example). Similar results have been obtained in [13] for  $U_q(\widehat{gl}(2|2))$ -vertex operators related to the four-dimensional modules  $W_z$  and  $W_z^*$  defined in section 2.2.

For two  $U_q(\widehat{gl}(2|2))$ -modules  $V^{(1)}$  and  $V^{(2)}$  with bases  $\{v^{(1)}\}_j$  and  $\{v^{(2)}\}_{j'}$ , the  $R$ -matrix  $R_{V^{(1)}V^{(2)}}(\frac{z}{w}) \in \text{End}(V_z^{(1)} \otimes V_w^{(2)})$  intertwines the action of  $U_q(\widehat{sl}(2|2))$  on the tensor product of the two evaluation modules  $V_z^{(1)}$  and  $V_w^{(2)}$ :

$$(77) \quad R_{V^{(1)}V^{(2)}}(\frac{z}{w}) \Delta(a) = \Delta'(a) R_{V^{(1)}V^{(2)}}(\frac{z}{w}) \quad \forall a \in U_q(\widehat{sl}(2|2))$$

where  $\Delta'(a) = P^{gr} \circ \Delta(a)$  with  $P^{gr}(x_1 \otimes x_2) = (-1)^{|x_1| \cdot |x_2|} x_2 \otimes x_1$ . It is convenient to introduce a second matrix  $\bar{R}_{V^{(1)}V^{(2)}}(\frac{z}{w})$  by

$$\begin{aligned}
 (78) \quad R_{V^{(1)}V^{(2)}}(\frac{z}{w}) (v_{j_1}^{(1)} \otimes v_{j_2}^{(2)}) &= \sum_{j_3, j_4} (-1)^{|v_{j_3}^{(1)}| \cdot |v_{j_4}^{(2)}|} \left( \bar{R}_{V^{(1)}V^{(2)}} \right)_{j_1, j_2}^{j_3, j_4} \left( \frac{z}{w} \right) v_{j_3}^{(1)} \otimes v_{j_4}^{(2)} \\
 \bar{R}_{V^{(1)}V^{(2)}}(\frac{z}{w}) (v_{j_1}^{(1)} \otimes v_{j_2}^{(2)}) &= \sum_{j_3, j_4} \left( \bar{R}_{V^{(1)}V^{(2)}} \right)_{j_1, j_2}^{j_3, j_4} \left( \frac{z}{w} \right) v_{j_3} \otimes v_{j_4}
 \end{aligned}$$

Uniqueness of the normalised vertex operators (70) provided, their commutation relation reads

$$(79) \quad \left( \Phi_{V_J}^{V^{(1)} V_K} \right)_{j_1} (z) \left( \Phi_{V_I}^{V^{(2)} V_J} \right)_{j_2} (w) = c\left(\frac{z}{w}\right) \cdot \sum_{j_3, j_4} \left( \bar{R}_{V^{(1)} V^{(2)}} \right)_{j_1, j_2}^{j_3, j_4} \left( \frac{z}{w} \right) \left( \Phi_{V_J}^{V^{(2)} V_K} \right)_{j_4} (w) \left( \Phi_{V_I}^{V^{(1)} V_J} \right)_{j_3} (z)$$

with a scalar function  $c\left(\frac{z}{w}\right)$ . The relation (79) follows from the argument given in chapter 6 of [2] for a pair of evaluation modules  $V_z^{(1)}$  and  $V_w^{(2)}$  with an intertwiner  $R_{V^{(1)} V^{(2)}}\left(\frac{z}{w}\right)$ . For quantum affine algebras, the normalisation of the  $R$ -matrix governing the analogous commutations relations has been determined via free boson realizations [2] or from the related two-point functions obtained as solutions of certain  $q$ -difference equations [18]. A free boson realization has also been used in [13] to fix the commutation relations of two  $U_q(\widehat{gl}(2|2))$ -vertex operators related to  $W_z$  or  $W_z^*$ .

The  $U'_q(\widehat{sl}(2|2))$ -module relevant to the configuration spaces  $\Omega_A^{(1)}$  and  $\Omega_B^{(1)}$  of the present vertex model is constructed from the infinite-dimensional, irreducible  $U_q(gl(2|2))$ -module  $V_{\bar{\Lambda}_2 + \bar{\Lambda}_4}$  with basis  $\{v_{j,t}\}_{0 \leq j \leq 3, t \in \mathbb{N}_0}$ . Its  $U_q(gl(2|2))$ -weights are

$$(80) \quad \begin{aligned} h_1 v_{j,t} &= -(t+1 - \delta_{j,2}) v_{j,t} & h_2 v_{j,t} &= (t+1 - \delta_{j,0}) v_{j,t} \\ h_3 v_{j,t} &= (t+1 - \delta_{j,2}) v_{j,t} & h_4 v_{j,t} &= -(t-1 + \delta_{j,0} + 2\delta_{j,1}) v_{j,t} \end{aligned}$$

A  $\mathbb{Z}_2$ -grading is defined by  $|v_{0,t}| = |v_{2,t}| = 1$  and  $|v_{1,t}| = |v_{3,t}| = 0 \ \forall t$ . The evaluation homomorphism found in [11] yields a  $U'_q(\widehat{sl}(2|2))$ -structure given by the action of  $h_1, h_2, h_3$  specified in (80) and

$$(81) \quad \begin{aligned} H_m^1 v_{j,t} &= -\frac{1}{m} q^{m(t+1-\delta_{j,2})} [m(t+1 - \delta_{j,2})] v_{j,t} \\ H_m^2 v_{j,t} &= \frac{1}{m} q^{m(t+\delta_{j,0}+2\delta_{j,1})} [m(t+1 - \delta_{j,0})] v_{j,t} \\ H_m^3 v_{j,t} &= -H_m^1 v_{j,t} \end{aligned}$$

for  $m \neq 0$  and

$$(82) \quad \begin{aligned} E_m^{1,-} v_{2,t} &= v_{1,t-1} & E_m^{1,+} v_{1,t} &= -[t+1] v_{2,t+1} \\ E_m^{1,-} v_{3,t} &= v_{0,t} & E_m^{1,+} v_{0,t} &= -[t+1] v_{3,t} \\ E_m^{2,-} v_{1,t} &= -q^{m(2t+3)} \frac{[t+1]}{[t+2]} v_{0,t+1} & E_m^{2,+} v_{0,t+1} &= -q^{m(2t+3)} [t+2] v_{1,t} \\ E_m^{2,-} v_{2,t} &= q^{m(2t+1)} v_{3,t} & E_m^{2,+} v_{3,t} &= q^{m(2t+1)} [t+1] v_{2,t} \\ E_m^{3,-} v_{0,t} &= -q^{2m(t+1)} v_{1,t} & E_m^{3,+} v_{1,t} &= -q^{2m(t+1)} [t+1] v_{0,t} \\ E_m^{3,-} v_{3,t} &= q^{2m(t+1)} v_{2,t+1} & E_m^{3,+} v_{2,t+1} &= q^{2m(t+1)} [t+1] v_{3,t} \end{aligned}$$

$\forall m \in \mathbb{Z}$ . The basis  $\{v_{j,t}\}_{0 \leq j \leq 3}$  coincides with the basis of the module  $V$  introduced in [1]. A reducible, infinite-dimensional  $U'_q(\widehat{sl}(2|2))$ -module  $\hat{V}$  with basis  $v_{1,-1} \cup \{v_{j,t}\}_{0 \leq j \leq 3, t \in \mathbb{N}_0}$  is defined by (80)-(82) and

$$(83) \quad \begin{aligned} h_l v_{1,-1} &= 0 & l &= 1, 2, 3, 4 \\ H_m^k v_{1,-1} &= 0 & k &= 1, 2, 3, m \neq 0 \end{aligned}$$



and

$$\begin{aligned}
 E_m^{1,-} v_{1,-1} &= E_m^{2,+} v_{1,-1} = E_m^{3,\pm} v_{1,-1} = 0 \\
 E_m^{1,+} v_{1,-1} &= \frac{[m]}{m} v_{2,0} \\
 E_m^{2,-} v_{1,-1} &= q^m \frac{[m]}{m} v_{0,0}
 \end{aligned}
 \tag{84}$$

$\forall m$ . Any action of Drinfeld generators of  $U'_q(\widehat{sl}(2|2))$  on  $\check{V}$  not listed in (80)-(84) vanishes. Specifying the action of all  $H_m^4$  on  $v_{j,t}$  for one pair  $j, t$  determines a  $U'_q(\widehat{gl}(2|2))$ -structure on  $\check{V}$ . The appropriate choice will be given below.

For shorter notation, the components  $\left(\Phi_{V_I}^{\check{V} V_J}\right)_{j,t}(z)$  may be written  $\Phi_{j,t}(z)$ . The intertwining property (73) with  $a = e_k$ ,  $k = 1, 2, 3$ , relates all components  $\left(\Phi_{V_I}^{\check{V} V_J}\right)_{j,t}(z)$  with  $j = 0, 1, 2, 3$  and  $t \geq 0$ :

$$\begin{aligned}
 [t+1]\Phi_{3,t}(z) &= q^{t+1} e_2 \Phi_{2,t}(z) + \Phi_{2,t}(z) e_2 \\
 [t+1]\Phi_{2,t+1}(z) &= -q^{t+1} e_3 \Phi_{3,t}(z) + \Phi_{3,t}(z) e_3 \\
 [t+1]\Phi_{0,t}(z) &= q^{-t-1} e_1 \Phi_{3,t}(z) - \Phi_{3,t}(z) e_1 \\
 [t+1]\Phi_{1,t}(z) &= -q^{-t-1} e_1 \Phi_{2,t+1}(z) - \Phi_{2,t+1}(z) e_1
 \end{aligned}
 \tag{85}$$

According to (74)-(76), the intertwining property (73) imposes the following conditions on the component  $\Phi_{2,0}(z)$ :

$$\begin{aligned}
 E_m^{k,-} \Phi_{2,0}(z) - (-1)^{|\Phi_{2,0}|} \Phi_{2,0}(z) E_m^{k,-} &= 0 \quad k = 1, 2, 3 \\
 E_m^{3,+} \Phi_{2,0}(z) - (-1)^{|\Phi_{2,0}|} \Phi_{2,0}(z) E_m^{3,+} &= 0
 \end{aligned}
 \tag{86}$$

$$E_m^{1,+} \Phi_{2,0}(z) - (-1)^{|\Phi_{2,0}|} \Phi_{2,0}(z) E_m^{1,+} = -(-1)^{|v_{2,0}|} q^m z^m \Phi_{1,-1}(z)
 \tag{87}$$

$\forall m$  and

$$\begin{aligned}
 [h_l, \Phi_{2,0}(z)] &= -(\delta_{l,2} + \delta_{l,4}) \Phi_{2,0}(z) \quad l = 1, 2, 3, 4 \\
 w^{-d} \Phi_{2,0}(z) w^d &= \Phi_{2,0}(wz) \\
 [H_{\pm n}^1, \Phi_{2,0}(z)] &= [H_{\pm n}^3, \Phi_{2,0}(z)] = 0 \\
 [H_n^2, \Phi_{2,0}(z)] &= -q^{\frac{1}{2}n} \frac{[n]}{n} z^n \Phi_{2,0}(z) \\
 [H_{-n}^2, \Phi_{2,0}(z)] &= -q^{-\frac{3}{2}n} \frac{[n]}{n} z^{-n} \Phi_{2,0}(z)
 \end{aligned}
 \tag{88}$$

for  $n > 0$ . The conditions (86) and (88) are fulfilled by

$$\begin{aligned}
 &\Phi_{2,0}(z) = \\
 &: \exp \left\{ \beta^1(qz) - i\varphi^1 - \varphi^2 - \ln(qz)(\varphi_0^1 - i\varphi_0^2) - i\pi\varphi_0^3 + C(\varphi_0^1 - i\varphi_0^2 - \varphi_0^3 + i\varphi_0^4) \right. \\
 &\quad \left. + \sum_{m \neq 0} C_m(\varphi_m^1 - i\varphi_m^2 - \varphi_m^3 + i\varphi_m^4) z^{-m} + \sum_{m \neq 0} \frac{1}{[m]} q^{-\frac{1}{2}|m|-m} (\varphi_m^3 - i\varphi_m^4) z^{-m} \right\} :
 \end{aligned}
 \tag{89}$$

and

$$(90) \quad \Phi_{1,-1}(z) =: \exp \left\{ C(\varphi_0^1 - i\varphi_0^2 - \varphi_0^3 + i\varphi_0^4) \right. \\ \left. + \sum_{m \neq 0} \left( C_m - \frac{1}{[m]} q^{-\frac{1}{2}|m|-m} \right) (\varphi_m^1 - i\varphi_m^2 - \varphi_m^3 + i\varphi_m^4) z^{-m} \right\} :$$

for  $C, C_m \in \mathbb{C}$ . Here the free boson realizations (5), (8) and (11) for the Drinfeld generators have been employed. The expressions (89) and (90) satisfy

$$(91) \quad \Phi_{2,0}(z)\Phi_{2,0}(w) = -\Phi_{2,0}(w)\Phi_{2,0}(z)$$

and

$$(92) \quad \Phi_{1,-1}(z)\Phi_{1,-1}(w) = \Phi_{1,-1}(w)\Phi_{1,-1}(z) \quad \Phi_{1,-1}(z)\Phi_{2,0}(w) = \Phi_{2,0}(w)\Phi_{1,-1}(z)$$

As required by (73),  $\Phi_{1,-1}(z)$  commutes with  $h_l$ ,  $H_{\pm n}^k$  and  $E_m^{k,\pm}$  with  $k = 1, 2, 3$ ,  $l = 1, 2, 3, 4$ ,  $n > 0$ ,  $m \neq 0$ . Due to (76), the property (73) is satisfied for  $a = H_m^4$  provided that

$$(93) \quad H_m^4 v_{2,0} = -\frac{[m]}{m} \left( 2q^{-m+\frac{1}{2}|m|} [m] C_{-m} + 1 \right) z^m v_{2,0} \quad m \neq 0$$

The components  $\Phi_{1,-1;m}$  and  $\Phi_{2,0;m}$  defined by  $\Phi_{1,-1}(z) = \sum_{m \in \mathbb{Z}} \Phi_{1,-1;m} z^{-m}$  and  $\Phi_{2,0}(z) = \sum_{m \in \mathbb{Z}} \Phi_{2,0;m} z^{-m}$  may be applied to the level-one modules  $\mathring{V}(\Lambda_0)$ ,  $\mathring{V}(\Lambda_1 + \Lambda_4)$ ,  $\mathring{V}(2\Lambda_2 - \Lambda_3 + \Lambda_4)$  or  $V(-s\Lambda_1 + (1+s)\Lambda_2 - (1+2s')\Lambda_4)$  introduced in section 3. Use of the realizations (36) and taking into account the property (31) reveals that the components  $\Phi_{1,-1;m}$  and  $\Phi_{2,0;m}$  map each of these level-one modules onto itself if

$$(94) \quad [m]C_m = q^{-\frac{1}{2}|m|-m} \quad \text{for } m < 0$$

This suggests the existence of intertwiners of the form (70) with  $V = \mathring{V}$  and  $V_I = V_J$  given by any of the four level-one modules. The validity of the entire set of equations (73) remains to be demonstrated.

**Conjecture 2.** *There exist type II vertex operators*

$$(95) \quad \Phi_{V_I}^{\mathring{V} V_I}(z) : V_I \longrightarrow \mathring{V}_z \otimes V_I \quad I = 0, 1, 3, 4$$

for  $V_0 = \mathring{V}(\Lambda_0)$ ,  $V_1 = \mathring{V}(\Lambda_1 + \Lambda_4)$ ,  $V_3 = \mathring{V}(2\Lambda_0 - \Lambda_3 + \Lambda_4)$  and  $V_4 = V(-s\Lambda_1 + (1+s)\Lambda_2 - (1+2s')\Lambda_4)$ . They can be normalised such that

$$(96) \quad \left( \Phi_{V_I}^{\mathring{V} V_I} \right)_{1,-1}(z) =: \exp \left\{ C(\varphi_0^1 - i\varphi_0^2 - \varphi_0^3 + i\varphi_0^4) \right. \\ \left. + \sum_{m > 0} \left( C_m - \frac{1}{[m]} q^{-\frac{1}{2}|m|-m} \right) (\varphi_m^1 - i\varphi_m^2 - \varphi_m^3 + i\varphi_m^4) z^{-m} \right\} :$$

with arbitrary complex numbers  $C$  and  $C_m$  with  $m > 0$ . A further component is given by

$$(97) \quad \left( \Phi_{V_I}^{\hat{V} V_I} \right)_{2,0}(z) = \\ : \exp \left\{ \beta^1(qz) - i\varphi^{1,+}(qz) - \varphi^{2,+}(qz) - i\pi\varphi_0^3 + C(\varphi_0^1 - i\varphi_0^2 - \varphi_0^3 + i\varphi_0^4) \right. \\ \left. + \sum_{m>0} (C_m - \frac{1}{[m]} q^{-\frac{1}{2}|m|-m}) (\varphi_m^1 - i\varphi_m^2 - \varphi_m^3 + i\varphi_m^4) z^{-m} \right\} :$$

All the remaining components  $\left( \Phi_{V_I}^{\hat{V} V_I} \right)_{j,t}(z)$ ,  $0 \leq j \leq 3$ ,  $t - \delta_{j,2} \geq 0$ , follow from (85) and (97). The vertex operators satisfy the commutation relation

$$(98) \quad \Phi_{V_I}^{\hat{V} V_I}(z) \Phi_{V_I}^{\hat{V} V_I}(w) = -\bar{R}_{\hat{V}\hat{V}} \left( \frac{z}{w} \right) \Phi_{V_I}^{\hat{V} V_I}(w) \Phi_{V_I}^{\hat{V} V_I}(z)$$

with the  $R$ -matrix  $R_{\hat{V}\hat{V}} \left( \frac{z}{w} \right) : \hat{V}_z \otimes \hat{V}_w \longrightarrow \hat{V}_z \otimes \hat{V}_w$  normalised by

$$(99) \quad \left( \bar{R}_{\hat{V}\hat{V}} \right)_{2,0;2,0}^{2,0;2,0} \left( \frac{z}{w} \right) = 1$$

The normalisation (99) stems from the relation (91). All  $R$ -matrix elements involving the index pair  $1, -1$  are listed by  $(\bar{R}_{\hat{V}\hat{V}})_{1,-1;1,-1}^{1,-1;1,-1}(z) = 1$  and  $(\bar{R}_{\hat{V}\hat{V}})_{1,-1;j,t}^{1,-1;j,t}(z) = (\bar{R}_{\hat{V}\hat{V}})_{j,t;1,-1}^{j,t;1,-1}(z) = 1 \forall j, t$ .

The action of  $H_m^4$  on  $v_{j,t}$  follows from the values of  $C_m$ . With

$$(100) \quad C = 0, \quad [m]C_m = q^{-\frac{|m|}{2}-m} \quad \forall m \neq 0$$

equation (93) yields

$$(101) \quad H_m^4 v_{1,-1} = 0 \\ H_m^4 v_{0,t} = -\frac{1}{m} q^{m(t+1)} [mt] v_{0,t} \quad H_m^4 v_{1,t} = -\frac{1}{m} q^{m(t+2)} [m(t+1)] v_{1,t} \\ H_m^4 v_{2,t} = \frac{1}{m} ([m] - q^{m(t+1)} [tm]) v_{2,t} \quad H_m^4 v_{3,t} = \frac{1}{m} ([m] - q^{m(t+1)} [tm]) v_{3,t}$$

$\forall m \neq 0$  and  $t \in \mathbb{N}_0$ . (100) is a convenient choice kept for the remainder of this section.

A similar analysis indicates the existence of type II vertex operators

$$(102) \quad \Phi_{V_I}^{\hat{V}^* V_I}(z) : V_I \longrightarrow \hat{V}_z^* \otimes V_I \quad I = 0, 1, 3, 4$$

for the dual module  $\hat{V}^*$ . In terms of the dual basis  $v_{1,-1}^* \cup \{v_{j,t}^*\}_{0 \leq j \leq 3, t \in \mathbb{N}_0}$  with  $\langle v_{j_1,t_1}^* | v_{j_2,t_2} \rangle = \delta_{j_1,j_2} \delta_{t_1,t_2}$ , the  $U'_q(\widehat{gl}(2|2))$ -structure on  $\hat{V}^*$  is introduced by

$$(103) \quad \langle av^* | v \rangle = (-1)^{|a| \cdot |v^*|} \langle v^* | S(a)v \rangle \quad \forall a \in U'_q(\widehat{gl}(2|2))$$

where  $S$  denotes the antipode. On  $U_q(\widehat{gl}(2|2))$ , the antipode is defined by

$$(104) \quad S(E_0^{k,+}) = -q^{-h_k} E_0^{k,+} \quad S(E_0^{k,-}) = -E_0^{k,-} q^{h_k} \quad S(h_l) = -h_l$$

for  $k = 1, 2, 3$  and  $l = 1, 2, 3, 4$ . Its definition on  $U_q(\widehat{gl}(2|2))$  can be found in [14].

For two vertex operators (102), the commutation relation reads

$$(105) \quad \Phi_{V_I}^{\hat{V}^* V_I}(z) \Phi_{V_I}^{\hat{V}^* V_I}(w) = -\bar{R}_{\hat{V}^* \hat{V}} \left( \frac{z}{w} \right) \Phi_{V_I}^{\hat{V}^* V_I}(w) \Phi_{V_I}^{\hat{V}^* V_I}(z)$$

In the following, the index pairs of the R-matrix elements will be indicated by  $j_i, t_i$  if associated with  $\check{V}$  and by  $j_i, t_i^*$  if associated to  $\check{V}^*$ . Then the subscript specifying  $V^{(1)}V^{(2)}$  in (78) and (79) can be skipped. The R-matrix elements of  $\bar{R}_{\check{V}^*\check{V}^*}(z)$  can be obtained from the matrix elements of  $\bar{R}_{\check{V}\check{V}}(z)$  by means of the relation

$$(106) \quad \bar{R}_{j_1, t_1^*; j_2, t_2^*}^{j_3, t_3^*; j_4, t_4^*}(z) = \bar{R}_{j_3, t_3; j_4, t_4}^{j_1, t_1; j_2, t_2}(z)$$

Explicite expressions for the matrix elements of  $\bar{R}_{\check{V}\check{V}}(z)$  are found in appendix B. In order to obtain well-defined limits for the R-matrix elements in the limit  $q \rightarrow 0$ , the basis of the modules  $V$  and  $V^*$  needs to be changed slightly. A new basis  $\{\tilde{v}_{j,t}\}_{0 \leq j \leq 3}$  or  $\{\tilde{v}_{j,t}^*\}_{0 \leq j \leq 3}$  is introduced by

$$(107) \quad \tilde{v}_{j,t} = q^{t(\alpha_j)} v_{j,t} \quad \tilde{v}_{j,t}^* = q^{-t(\alpha_j)} v_{j,t}^*$$

with  $\alpha_j = \frac{1}{2} + \delta_{j,1} - \delta_{j,2}$ . Then the expressions collected in formulae (161)-(185) yield

$$(108) \quad \lim_{q \rightarrow 0} \tilde{R}_{j_1, t_1; j_2, t_2}^{j_3, t_3; j_4, t_4}(z) = \delta_{j_1, j_4} \delta_{j_2, j_3} \delta_{t_1, t_4} \delta_{t_2, t_3} z^{t_2 + \theta_{j_2, j_1}}$$

Here  $\tilde{R}_{j_1, t_1; j_2, t_2}^{j_3, t_3; j_4, t_4}(z)$  denotes the R-matrix elements in the basis  $\{\tilde{v}_{j,t}\}_{0 \leq j \leq 3}$  as given by (187) and  $\theta_{j_2, j_1}$  is defined by (54). Changing the basis according to (107) does not affect the expression (97). With respect to the basis  $\{\tilde{v}_{j,t}\}_{0 \leq j \leq 3, t \in \mathbb{N}_0}$  or  $\{\tilde{v}_{j,t}^*\}_{0 \leq j \leq 3, t \in \mathbb{N}_0}$ , the commutation relations (79) for  $V = \check{V}$  or  $V = \check{V}^*$  with  $0 \leq j \leq 3, t \in \mathbb{N}_0$  may be viewed as equations for objects with small  $q$  expansions  $\left(\Phi_{V_I}^{\check{V} V_I}\right)_{j,t}(z^{-1}) = \phi_{j,t}(z) + O(q)$  or  $\left(\Phi_{V_I}^{\check{V}^* V_I}\right)_{j,t}(z^{-1}) = \phi_{j,t}^*(z) + O(q)$ . Then in the limit of vanishing  $q$  the commutation relations reads

$$(109) \quad \begin{aligned} \phi_{j_1, t_1}(z) \phi_{j_2, t_2}(w) &= - \left(\frac{z}{w}\right)^{-t_2 - \theta_{j_2, j_1}} \phi_{j_1, t_1}(w) \phi_{j_2, t_2}(z) \\ \phi_{j_1, t_1}^*(z) \phi_{j_2, t_2}^*(w) &= - \left(\frac{z}{w}\right)^{-t_1 - \theta_{j_1, j_2}} \phi_{j_1, t_1}^*(w) \phi_{j_2, t_2}^*(z) \end{aligned}$$

With the mode expansions  $\phi_{j,t}(z) = \sum_{m \in \mathbb{Z}} \phi_{j,t;m} z^{-m}$ ,  $\phi_{j,t}^*(z) = \sum_{m \in \mathbb{Z}} \phi_{j,t;m}^* z^{-m}$ , the relations (109) yield the defining equations (53), (64) for the creation algebras given in section 5:

$$(110) \quad \begin{aligned} \phi_{j_1, t_1; m_1} \phi_{j_2, t_2; m_2} &= - \phi_{j_1, t_1; m_2 + t_2 + \theta_{j_2, j_1}} \phi_{j_2, t_2; m_1 - t_2 - \theta_{j_2, j_1}} \\ \phi_{j_1, t_1; m_1}^* \phi_{j_2, t_2; m_2}^* &= - \phi_{j_1, t_1; m_2 + t_1 + \theta_{j_1, j_2}}^* \phi_{j_2, t_2; m_1 - t_1 - \theta_{j_1, j_2}}^* \end{aligned}$$

The properties (59) and (68) follow from (88).  $\Phi_{1,-1;0}^*$  ( $\Phi_{1,-1;0}$ ) can be viewed as the component corresponding to the unique semi-standard super tableau in  $\mathcal{B}_{1,1}^h$  ( $\mathcal{B}_{1,1}^v$ ) associated with the border strip consisting of a single row (column).

A free boson realization for  $\Phi_{V_I}^{\check{V}^* V_I}(z)$  is obtained employing the coproduct of the antipode of the  $U_q(\widehat{gl}(2|2))$ -generators. Replacing  $q$  by  $q^{-1}$  on the rhs of (14) gives a free boson realization of  $-q^m S(H_m^l)$ . On the lhs of (9),  $X^{1,\pm}(z)$  is replaced by  $X^{2,\pm}(z)$  and vice versa. Then substitution of  $q$  by  $q^{-1}$  on the rhs of (8) yields a suitable free boson realization of  $-q^{\pm h_k} S(E^{k,\pm}(q^{-1}z))$ . Coproduct and antipode of the algebra satisfy  $\Delta S(a) = (S \otimes S) \Delta'(a)$ . Due to this property, the coproduct formulae (76) give sufficient information to provide a free boson expression for

$\left(\Phi_{V_I}^{\hat{V}^* V_I}\right)_{2,0}(z)$  analogous to (97). The result yields the commutation relation

$$(111) \quad \left(\Phi_{V_I}^{\hat{V}^* V_I}\right)_{2,0}(z) \left(\Phi_{V_I}^{\hat{V}^* V_I}\right)_{2,0}(w) = - \left(\Phi_{V_I}^{\hat{V}^* V_I}\right)_{2,0}(w) \left(\Phi_{V_I}^{\hat{V}^* V_I}\right)_{2,0}(z)$$

Together with (106), this leads to (105).

The relation with the creation algebras suggests that the particular vertex operators discussed above create eigenstates of the row-to-row transfer matrix.

**Conjecture 3.** *The type II vertex operators  $\Phi_{V_I}^{\hat{V}^* V_I}(z)$  and  $\Phi_{V_I}^{\hat{V}^* V_I}(z)$  provide a part of the quasi-particle structure of the vertex model.*

Further details will be given together with a more complete account in a forthcoming publication.

The limiting behaviour of the R-matrix elements  $\tilde{R}_{j_1, t_1; j_2, t_2}^{j_3, t_3; j_4, t_4}(z)$  specified in (108) reflects the eigenvalue structure of the intertwiner  $R_{VV}(z) = P^{gr} R_{VV}(z)$ . The decomposition of the tensor product  $V \otimes V$  into irreducible  $U_q(\widehat{gl}(2|2))$ -modules is given by

$$(112) \quad V \otimes V = V_{2\bar{\Lambda}_2 + 2\bar{\Lambda}_4} \oplus \left( \bigoplus_{n=0}^{\infty} V_{-(n+1)\bar{\Lambda}_1 + (n+2)\bar{\Lambda}_2 + (n+1)\bar{\Lambda}_3 + (2-n)\bar{\Lambda}_4} \right)$$

Using the explicit expressions for the matrix elements listed in appendix B, the eigenvalues of  $\tilde{R}_{VV}(z)$  are readily evaluated. On any vector  $\sum_{j_1, j_2, t_1, t_2} \varkappa_{j_1, t_1; j_2, t_2} v_{j_1, t_1} \otimes v_{j_2, t_2}$  contained in the component  $V_{-(n+1)\bar{\Lambda}_1 + (n+2)\bar{\Lambda}_2 + (n+1)\bar{\Lambda}_3 + (2-n)\bar{\Lambda}_4}$ , the linear operator  $\tilde{R}_{VV}(z)$  acts as the scalar

$$(113) \quad r_n(z) = \frac{z - q^{2(n+1)}}{1 - q^{2(n+1)}z} \cdot \dots \cdot \frac{z - q^4}{1 - q^4z} \frac{z - q^2}{1 - q^2z}$$

Here the normalisation is chosen such that  $\tilde{R}_{VV}(z)$  acts as the unit on the component  $V_{2\bar{\Lambda}_2 + 2\bar{\Lambda}_4}$ . The  $U_q(\widehat{gl}(2|2))$ -weight space of the component  $V_{-(n+1)\bar{\Lambda}_1 + (n+2)\bar{\Lambda}_2 + (n+1)\bar{\Lambda}_3 + (2-n)\bar{\Lambda}_4}$  is in one-to-one correspondence with the pairs  $(j_1, t_1; j_2, t_2)$  attributed to the power  $z^n$  by (108). Hence the  $U_q(\widehat{gl}(2|2))$ -weights attributed to all normal forms  $\phi_{j_2, t_2; m+n+1} \phi_{j_1, t_1; m}$  with  $n > 0$  and  $m$  fixed exactly corresponds to the  $U_q(\widehat{gl}(2|2))$ -weights of the components  $V_{2\bar{\Lambda}_2 + 2\bar{\Lambda}_4}$  and  $V_{-(n'+1)\bar{\Lambda}_1 + (n'+2)\bar{\Lambda}_2 + (n'+1)\bar{\Lambda}_3 + (2-n')\bar{\Lambda}_4}$  with  $n' < n$ . The  $U_q(\widehat{gl}(2|2))$ -weights of  $\phi_{j_r, t_r; m+r-1} \dots \phi_{j_2, t_2; t_2} \phi_{j_1, t_1; m_1}$  correspond to the weights of  $V_{r\bar{\Lambda}_2 + r\bar{\Lambda}_4}$ . Generally, a finite set of infinite-dimensional  $U_q(\widehat{gl}(2|2))$ -modules can be associated with each infinite vertical or horizontal border strip with parameters  $r_0, (p_1, \dots, p_{R-1}; \bar{p}_1, \dots, \bar{p}_R)$  such that the weights found in the modules coincide with the weights of the normal forms in the set  $B(p_1, \dots, p_{R-1}; \bar{p}_1, \dots, \bar{p}_R; r_0)$ . Analogous statements apply to horizontal border strips.

In contrast, the infinite configuration space of the vertex model does not seem to indicate the structure of  $V \otimes V^*$  or  $V^* \otimes V$  in an analogous way. Within a certain region of the spectral parameter  $z$ , the action of the R-matrix on the tensor product  $V \otimes V^*$  is analysed in the following section.

## 7. THE MIXED CASE

The description of the space of infinite configurations outlined in sections 4 and 5 is achieved by means of two sets of normal forms. Each set is provided by a creation algebra and applies to the description of one space of components. The two creation

algebras can be taken independent. As far as described by the previous section, the limiting procedure connecting the type II vertex operators to the creation algebras treats both pairs  $(\Phi_{V_{I_1}}^{V V_{J_1}}(z), \mathcal{A})$  and  $(\Phi_{V_{I_2}}^{V^* V_{J_2}}(w), \mathcal{A}^*)$  separately. This raises the questions whether combining both vertex operators for the description of the entire configuration space is compatible with a description by two independent creation algebras. Assuming that  $V_z \otimes V_w^*$  and  $V_w^* \otimes V_z$  are intertwined by  $R_{VV^*}(\frac{z}{w})$  according to (77), the commutation relations involving both types of vertex operators can be written in the form (79) with  $V^{(1)} = V$  and  $V^{(2)} = V^*$ . However, this choice of evaluation modules does not allow for the introduction of an intertwiner in the general sense. Though the intertwining conditions (77) can be solved formally for the single R-matrix elements (78), the corresponding action of the R-matrix on the  $U_q(gl(2|2))$ -irreducible components of  $V \otimes V^*$  is well defined only for  $|\frac{q^2 w}{z}| < 1$ . In the following, it will be shown that within this region, this action does not depend on the irreducible component in the limit  $q \rightarrow 0$ .

In certain cases, consideration of single matrix elements of  $\check{R}_{V^{(1)}V^{(2)}}(z) = P^{gr} R_{V^{(1)}V^{(2)}}(z)$  is less instructive in the case  $V^{(1)} \neq V^{(2)}$ . The normalisation of a vector in an irreducible component of  $V^{(1)} \otimes V^{(2)}$  can be chosen independently of the normalisation of a vector in the corresponding component of  $V^{(2)} \otimes V^{(1)}$ . Choices of the normalisations depend on the purpose, e.g. taking a certain limit, and is not apparent from a single matrix element. In fact, it seems hard to extract some formal limit of vanishing  $q$  from the equations (79) with  $V^{(1)} = V$  and  $V^{(2)} = V^*$ .

The decomposition of  $V \otimes V^*$  can be found studying the action of the quadratic Casimir  $C^{(2)}$  on the tensor products.

The operator  $C^{(2)}$  given by

$$\begin{aligned}
 (114) \quad & \frac{C^{(2)}}{(q - q^{-1})^2} = \\
 & - E_0^{1,-} E_0^{1,+} q^{h_2+h_3+h_4} - q^{-2} E_0^{2,-} E_0^{2,+} q^{-h_1+h_3+h_4} - E_0^{3,-} E_0^{3,+} q^{-h_1-h_2+h_4} \\
 & + q^{-1} [E_0^{1,-}, E_0^{2,-}]_{q^{-1}} [E_0^{1,+}, E_0^{2,+}]_q q^{h_3+h_4} - q^{-1} [E_0^{2,-}, E_0^{3,-}]_{q^{-1}} [E_0^{2,+}, E_0^{3,+}]_q q^{-h_1+h_4} \\
 & - (E_0^{3,-} [E_0^{1,-}, E_0^{2,-}]_{q^{-1}} - q^{-1} [E_0^{1,-}, E_0^{2,-}]_{q^{-1}} E_0^{3,-}) \cdot \\
 & \quad (E_0^{3,+} [E_0^{1,+}, E_0^{2,+}]_q - q [E_0^{1,+}, E_0^{2,+}]_q E_0^{3,+}) q^{h_4} \\
 & + \frac{[2]}{(q - q^{-1})^2} - \frac{q^{h_4-1}}{(q - q^{-1})^2} (q^{h_1+h_2+h_3} - q^{-h_1+h_2+h_3} + q^{-h_1-h_2+h_3} - q^{-h_1-h_2-h_3})
 \end{aligned}$$

is a central element of  $U_q(gl(2|2))$ . Here the  $q$ -deformed commutators are defined by

$$\begin{aligned}
 (115) \quad & [E_0^{i,+}, E_0^{i+1,+}]_q = E_0^{i,+} E_0^{i+1,+} + q^{(-1)^{i+1}} E_0^{i+1,+} E_0^{i,+} \\
 & [E_0^{i,-}, E_0^{i+1,-}]_{q^{-1}} = E_0^{i,-} E_0^{i+1,-} + q^{(-1)^i} E_0^{i+1,-} E_0^{i,-}
 \end{aligned}$$

Generalised eigenvectors of  $\Delta(C^{(2)})$  with eigenvalue  $2y$  are provided by

$$(116) \quad \sum_{t=0}^{\infty} q^{t(2r-1)} \frac{(q^2; q^2)_{t+1} (q^2; q^2)_r}{(q^2; q^2)_{t+r} (q^2; q^2)_1} \tilde{P}_t(y; q^{2r+1}, q, q|q^2) v_{2,t+r} \otimes v_{0,t}^* \quad r \geq 0$$

$$(117) \quad \sum_{t=0}^{\infty} q^{-t(2r+1)} \frac{(q^2; q^2)_{t+r+1}}{(q^2; q^2)_t (q^2; q^2)_{r+1}} \tilde{P}_t(y; q^{2r+1}, q, q|q^2) v_{2,t} \otimes v_{0,t+r}^* \quad r > 0$$

and

$$(118) \quad \sum_{t=0}^{\infty} q^{t(2r-1)} \frac{(q^2; q^2)_{t+1} (q^2; q^2)_r}{(q^2; q^2)_{t+r} (q^2; q^2)_1} \tilde{P}_t(y; q^{-2r-1}, q^{-1}, q^{-1}|q^{-2}) v_{0,t}^* \otimes v_{2,t+r} \quad r \geq 0$$

$$(119) \quad \sum_{t=0}^{\infty} q^{-t(2r+1)} \frac{(q^2; q^2)_{t+r+1}}{(q^2; q^2)_t (q^2; q^2)_{r+1}} \tilde{P}_t(y; q^{-2r-1}, q^{-1}, q^{-1}|q^{-2}) v_{0,t+r}^* \otimes v_{2,t} \quad r > 0$$

where  $r \geq 0$ ,  $(q^2; q^2)_n = (1 - q^2)(1 - q^4) \dots (1 - q^{2n})$  and  $\tilde{P}_t(y; a, c, d|q^2) \equiv \tilde{P}_t(y)$  denote orthonormal continuous dual Hahn polynomials in base  $q^2$  [19]. They satisfy the recurrence relation

$$(120) \quad 2y\tilde{P}_t(y) = a_t\tilde{P}_{t+1}(y) + b_t\tilde{P}_t(y) + a_{t-1}\tilde{P}_{t-1}(y)$$

where

$$(121) \quad \begin{aligned} a_t &= \sqrt{(1 - q^{2(t+1)})(1 - acq^{2t})(1 - adq^{2t})(1 - cdq^{2t})} \\ b_t &= a + a^{-1} - a^{-1}(1 - acq^{2t})(1 - adq^{2t}) - a(1 - q^{2t})(1 - cdq^{2(t-1)}) \end{aligned}$$

and  $t \geq -1$ ,  $\tilde{P}_0(y) = 1$  and  $\tilde{P}_{-1}(y) = 0$ . Replacing  $q$  by  $q^{-1}$  in the coefficients (121) yields the recursion relations for the polynomials  $\tilde{P}_t(y; a, c, d|q^{-2})$ . They are supplemented by the initial condition  $\tilde{P}_{-1}(y; a, c, d|q^{-2}) = 0$  and a convenient choice of the value of  $\tilde{P}_0(y; a, c, d|q^{-2})$  to be specified below. The polynomials  $\tilde{P}_t(y; a, c, d|q^2)$  are orthonormal with respect to the measure  $dm(\cdot; a, 0, c, d|q^2)$ . The measure  $dm(\cdot; a, b, c, d|q^2)$  is defined by

$$(122) \quad \int_{\mathbb{R}} f(y) dm(y; a, b, c, d|q^2) = \int_0^\pi f(\cos \theta) w(\cos \theta) d\theta + \sum_k f(y_k) w_k$$

with

$$\begin{aligned} w(\cos \theta) &= w(\cos \theta; a, b, c, d|q^2) \\ &= \frac{1}{2\pi} \frac{(q^2, ab, ac, ad, bc, bd, cd; q^2)_\infty}{(abcd; q^2)_\infty} \frac{(e^{2i\theta}, e^{-2i\theta}; q^2)_\infty}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta}; q^2)_\infty} \end{aligned}$$

and  $y_k = \frac{1}{2}(eq^{2k} + e^{-1}q^{-2k})$  for  $e$  any of the parameters  $a, b, c, d$ . The sum includes all  $k \in \mathbb{N}_0$  with the property  $|eq^{2k}| > 1$ . For  $e = b$ , the coefficients are given by

$$(123) \quad w_k = w_k(b; a, c, d|q^2) = \frac{1 - b^2 q^{4k}}{1 - b^2} \frac{(b^{-2}, ac, ad, cd; q^2)_\infty}{(a/b, c/b, d/b, abcd; q^2)_\infty} \frac{(b^2, ab, bc, bd; q^2)_k}{(q^2, q^2 b/a, q^2 b/c, q^2 b/d; q^2)_k} \left( \frac{q^2}{abcd} \right)^k$$

Throughout the following, the parameters  $a, c, d$  satisfy the condition  $\max(|a|, |c|, |d|) < 1$ . Thus the measure  $dm(\cdot; a, 0, c, d|q^2)$  contains only the continuous part. In (122) and (123), the notation for the  $q$ -shifted factorials is adopted from [20]:

$$(124) \quad (a_1, \dots, a_s; q^2)_\infty = (a_1; q^2)_\infty \dots (a_s; q^2)_\infty, \quad (a; q^2)_\infty = \prod_{j=0}^{\infty} (1 - aq^{2j})$$

$$(125) \quad (a_1, \dots, a_s; q^2)_n = (a_1; q^2)_n \dots (a_s; q^2)_n, \quad (a; q^2)_n = \prod_{j=0}^{n-1} (1 - aq^{2j})$$

$$(126) \quad (a_1, \dots, a_s; q^2)_0 = 1$$

The measure defined in (122) provides an orthogonality measure for the Askey-Wilson polynomials  $p_t(y; a, b, c, d|q^2)$  introduced by

$$(127) \quad p_t(y; a, b, c, d|q^2) = a^{-t} (ab, ac, ad; q^2)_t {}_4\phi_3(q^{-2t}, abcdq^{2(t-1)}, a\gamma, a\gamma^{-1}; ab, ac, ad; q^2, q^2)$$

with  $y = \frac{1}{2}(\gamma + \gamma^{-1})$ . Here the basic hypergeometric series [20] is defined by

$$(128) \quad {}_{n+1}\phi_n(a_1, a_1, \dots, a_{n+1}; b_1, \dots, b_n; q^2, \alpha) = \sum_{t=0}^{\infty} \frac{(a_1, a_2, \dots, a_{n+1}; q^2)_t}{(q^2, b_1, \dots, b_n; q^2)_t} \alpha^t$$

Due to Sears' transformation ([20], eq. 2.10.4), the polynomials  $p_t(y; a, b, c, d|q^2)$  are symmetric in  $a, b, c, d$ . The orthonormal continuous dual  $q^2$ -Hahn polynomials and the Askey-Wilson polynomials are related by

$$(129) \quad \tilde{P}_t(y; a, b, c, d|q^2) = \frac{1}{(q^2, ac; q^2)_t} p_t(y; a, 0, c, d|q^2)$$

The formal intertwining conditions on the R-matrix elements introduced on  $V_z \otimes V_w^*$  or  $V_z^* \otimes V_w$  are given by (77) and (78) with  $V^{(1)} = V$ ,  $V^{(2)} = V^*$  or  $V^{(1)} = V^*$ ,  $V^{(2)} = V$ . With the  $U_q(gl(2|2))$ -weight structure of  $V$  and  $V^*$  specified by (80) and (103), these conditions imply

$$(130) \quad \begin{aligned} \check{R}_{V^*V^*}\left(\frac{z}{w}\right)(v_{2,t_1} \otimes v_{0,t_2}^*) &= \sum_{t_3, t_4} \check{R}_{2,t_1; 0, t_2^*}^{0, t_4^*; 2, t_3}\left(\frac{z}{w}\right) v_{0, t_4}^* \otimes v_{2, t_3} \\ \check{R}_{V^*V}\left(\frac{w}{z}\right)(v_{0, t_2}^* \otimes v_{2, t_1}) &= \sum_{t_3, t_4} \check{R}_{0, t_2^*; 2, t_1}^{2, t_3; 0, t_4^*}\left(\frac{w}{z}\right) v_{2, t_3} \otimes v_{0, t_4}^* \end{aligned}$$

where  $t_1 - t_2 = t_3 - t_4$ . Equations (157) and (190) imply

$$(131) \quad \begin{aligned} \check{R}_{0, t_2^*; 2, t_2+r}^{2, t_1+r; 0, t_1^*}(z) &= q^{2(t_1-t_2)(2r-1)} \left( \frac{(q^2; q^2)_{t_2+r} (q^2; q^2)_{t_1+1}}{(q^2; q^2)_{t_1+r} (q^2; q^2)_{t_2+1}} \right)^2 \check{R}_{2, t_1+r; 0, t_1^*}^{0, t_2^*; 2, t_2+r}(z) \\ \check{R}_{0, t_2+r^*; 2, t_2}^{2, t_1; 0, t_1+r^*}(z) &= q^{2(t_2-t_1)(2r+1)} \left( \frac{(q^2; q^2)_{t_1+r+1} (q^2; q^2)_{t_2}}{(q^2; q^2)_{t_2+r+1} (q^2; q^2)_{t_1}} \right)^2 \check{R}_{2, t_1; 0, t_1+r^*}^{0, t_2+r^*; 2, t_2}(z) \end{aligned}$$

The matrix elements on the rhs of (130) may be viewed as the  $\check{R}$ -matrix elements associated with two evaluation modules of  $U_q(\widehat{sl}(2))$ . A convenient choice of



$U_q(\widehat{sl}(2))$ -generators is

$$\begin{aligned}
 \hat{h}_1 &= h_2 + h_3 \\
 \hat{h}_0 &= h_0 + h_1 \\
 \hat{e}_1 &= (E_0^{3,-} E_0^{2,-} + q^{-1} E_0^{2,-} E_0^{3,-}) q^{h_2+h_3} \\
 \hat{f}_1 &= -q^{-h_2-h_3} (E_0^{2,+} E_0^{3,+} + q E_0^{3,+} E_0^{2,+}) \\
 S(\hat{e}_0) &= -q^{h_2+h_3} (E_0^{2,+} E_{-1}^{3,+} + q^{-1} E_{-1}^{3,+} E_0^{2,+}) \\
 S(\hat{f}_0) &= (E_1^{3,-} E_0^{2,-} + q E_0^{2,-} E_1^{3,-}) q^{-h_2-h_3}
 \end{aligned} \tag{132}$$

The defining relations obeyed by the generators  $\hat{e}_0, \hat{e}_1, \hat{f}_0, \hat{f}_1, \hat{h}_0, \hat{h}_1$  (see [2], for example) are inherited from the defining relations of  $U_q(\widehat{sl}(2|2))$  [1]. Two infinite-dimensional  $U_q(\widehat{sl}(2))$ -modules  $\hat{V}$  and  $\hat{V}^*$  with basis  $\{\hat{v}_j\}_{j \in \mathbb{N}_0}$  and  $\{\hat{v}_j^*\}_{j \in \mathbb{N}_0}$  are introduced by

$$\hat{v}_t = q^{-\frac{1}{2}t(2t+1)} \frac{(q^2; q^2)_{t+1}}{(1-q^2)^{t+1}} v_{0,t}^*, \quad \hat{v}_t^* = q^{\frac{1}{2}t(2t-1)} \frac{(1-q^2)^t}{(q^2; q^2)_t} v_{2,t} \tag{133}$$

Then (132) and (80), (82) yield  $U_q(\widehat{sl}(2))$ -structures on  $\hat{V}$  and  $\hat{V}^*$ . With these structures,  $\{v_{2,t} \otimes z^m\}_{t \in \mathbb{N}_0, m \in \mathbb{Z}}$  and  $\{v_{0,t}^* \otimes w^m\}_{t \in \mathbb{N}_0, m \in \mathbb{Z}}$  can be regarded as  $U_q(\widehat{sl}(2))$ -evaluation modules  $\hat{V}_{z'}$  and  $\hat{V}_{w'}^*$ . With the definitions of the  $U_q(\widehat{sl}(2))$ -structures on  $\hat{V}_{z'}$  and  $\hat{V}_{w'}^*$  chosen analogous to (71), the spectral parameters are related by  $z' = q^{-3}z^{-1}$ ,  $w' = q^{-3}w^{-1}$ . The  $U_q(\widehat{sl}(2))$ -structures of  $\hat{V}$  and  $\hat{V}^*$  read

$$\begin{aligned}
 \hat{e}_0 \hat{v}_t &= q^{t-\frac{3}{2}} [t+1] \hat{v}_{t+1} & \hat{e}_0 \hat{v}_t^* &= -q^{-t-\frac{5}{2}} [t] \hat{v}_{t-1}^* \\
 \hat{f}_0 \hat{v}_t &= -q^{-t+\frac{5}{2}} [t] \hat{v}_{t-1} & \hat{f}_0 \hat{v}_t^* &= q^{t+\frac{7}{2}} [t+1] \hat{v}_{t+1}^* \\
 \hat{h}_0 \hat{v}_t &= (2t+1) \hat{v}_t & \hat{h}_0 \hat{v}_t^* &= -(2t+1) \hat{v}_t^* \\
 \hat{e}_1 \hat{v}_t &= -q^{-t-\frac{1}{2}} [t] \hat{v}_{t-1} & \hat{e}_1 \hat{v}_t^* &= q^{t+\frac{1}{2}} [t+1] \hat{v}_{t+1}^* \\
 \hat{f}_1 \hat{v}_t &= q^{t+\frac{3}{2}} [t+1] \hat{v}_{t+1} & \hat{f}_1 \hat{v}_t^* &= -q^{-t+\frac{1}{2}} [t] \hat{v}_{t-1}^* \\
 \hat{h}_1 \hat{v}_t &= -(2t+1) \hat{v}_t & \hat{h}_1 \hat{v}_t^* &= (2t+1) \hat{v}_t^*
 \end{aligned} \tag{134}$$

Apart from simple factors depending on  $r$ , the vectors (116), (118) can be rewritten by

$$\sum_{t=0}^{\infty} \tilde{P}_t(y; q^{2r+1}, q, q|q^2) \hat{v}_t^* \otimes \hat{v}_{t+r}, \quad \sum_{t=0}^{\infty} \tilde{P}_t(y; q^{2r+1}, q, q|q^2) \hat{v}_{t+r}^* \otimes \hat{v}_t \tag{135}$$

and

$$\sum_{t=0}^{\infty} \tilde{P}_t(y; q^{-2r-1}, q^{-1}, q^{-1}|q^{-2}) \hat{v}_{t+r} \otimes \hat{v}_t^*, \quad \sum_{t=0}^{\infty} \tilde{P}_t(y; q^{-2r-1}, q^{-1}, q^{-1}|q^{-2}) \hat{v}_t \otimes \hat{v}_{t+r}^* \tag{136}$$

They are generalised eigenvectors of the Casimir element of  $U_q(\widehat{sl}(2))$  given by

$$C = (q - q^{-1})^2 \hat{f}_1 \hat{e}_1 + q^{\hat{h}_1+1} + q^{-\hat{h}_1-1} \tag{137}$$

with eigenvalue  $2y$ . For fixed  $y = \cos \theta$ , the vectors (135) with  $r = 0, 1, 2, \dots$  constitute an infinite-dimensional  $U_q(\widehat{sl}(2))$ -module  $\hat{V}^{(y)}$  corresponding to the principal unitary series representation  $\pi_{\rho,0}^P$  of  $U_q(\widehat{su}(1,1))$  with  $\cos \theta = \frac{1}{2}(q^{2i\rho} + q^{-2i\rho})$  [21],

[22]. Each representation  $\pi_{\rho,0}^P$  is irreducible [23]. The tensor product  $\hat{V}^* \otimes \hat{V}$  decomposes into an integral over of all modules  $\hat{V}^{(\cos \theta)}$  with  $0 \leq \theta \leq \pi$  [21], [22].

The vectors in (116) have the  $U_q(gl(2|2))$ -weights  $(-r+1, r+1, r-1, -r+1)$  with  $r \in \mathbb{Z}$ . Acting with the coproducts of the  $U_q(gl(2|2))$ -generators on them produces further generalised eigenvectors of the quadratic Casimir  $C^{(2)}$ . Taking into account the defining relations of  $U_q(gl(2|2))$  (see [1]) and the explicit expression (114) for  $C^{(2)}$ , the set of all resulting linear independent vectors for fixed  $y$  can be specified. They provide an irreducible  $U_q(gl(2|2))$ -module  $V^{(y)}$ . The weights of the vectors composing  $V^{(y)}$  are listed by

$$(138) \quad \begin{aligned} &(-r+1, r+1, r-1, -r+1), (-r+1, r-1, r-1, -r+3), \\ &(-r+1, r-3, r-1, -r+1), (-r+1, r-1, r-1, -r-1), \\ &(-r+1, r, r-1, -r+2)_2, (-r+1, r-2, r-1, -r+2)_2, \\ &(-r+1, r-2, r-1, -r)_2, (-r+1, r, r-1, -r)_2, \\ &(-r+1, r-1, r-1, -r+1)_4. \end{aligned}$$

Here the subscripts denotes the multiplicity of generalised eigenvectors at the given weight.

A decomposition of the tensor product  $V \otimes V^*$  contains all  $U_q(gl(2|2))$ -modules  $V^{(\cos \theta)}$  with  $0 \leq \theta \leq \pi$  since the eigenvectors related to the first set of weights in (138) can be regarded as the set of vectors spanning the  $U_q(sl(2))$ -module  $\hat{V}^{(y)}$ . Inspection of  $U_q(gl(2|2))$ -weight structure of  $V \otimes V^*$  and the list (138) shows that no further modules occur in the decomposition of  $V \otimes V^*$ .

The intertwining condition (77) should apply with the particular choice  $a = C^{(2)}$ . Thus, if the action of  $\tilde{R}_{VV^*}(z)$  is well defined on the irreducible components of  $V \otimes V^*$ , it is expected to take the form

$$(139) \quad \begin{aligned} &\tilde{R}_{VV^*}(z) \left( \sum_{t=0}^{\infty} q^{-t(2r+1)} \frac{(q^2; q^2)_{t+r+1}}{(q^2; q^2)_t (q^2; q^2)_{r+1}} \tilde{P}_t(y; q^{2r+1}, q, q|q^2) v_{2,t} \otimes v_{0,t+r}^* \right) = \\ &r(y, z) \sum_{t=0}^{\infty} q^{-t(2r+1)} \frac{(q^2; q^2)_{t+r+1}}{(q^2; q^2)_t (q^2; q^2)_{r+1}} \tilde{P}_t(y; q^{-2r-1}, q^{-1}, q^{-1}|q^{-2}) v_{0,t+r}^* \otimes v_{2,t} \end{aligned}$$

or

$$(140) \quad \begin{aligned} &\tilde{R}_{VV^*}(z) \left( \sum_{t=0}^{\infty} q^{t(2r-1)} \frac{(q^2; q^2)_{t+1} (q^2; q^2)_r}{(q^2; q^2)_{t+r} (q^2; q^2)_1} \tilde{P}_t(y; q^{2r+1}, q, q|q^2) v_{2,t+r} \otimes v_{0,t}^* \right) = \\ &r(y, z) \sum_{t=0}^{\infty} q^{t(2r-1)} \frac{(q^2; q^2)_{t+1} (q^2; q^2)_r}{(q^2; q^2)_{t+r} (q^2; q^2)_1} \tilde{P}_t(y; q^{-2r-1}, q^{-1}, q^{-1}|q^{-2}) v_{0,t}^* \otimes v_{2,t+r} \end{aligned}$$

with  $|y| = 1$  and a suitable normalisation of the polynomials  $\tilde{P}_t(y; q^{-2r-1}, q^{-1}, q^{-1}|q^{-2})$ . Within a certain range of spectral parameters  $z$ , the function  $r(y, z)$  can be expressed as an infinite sum over elements of  $R_{VV}(q^2 z^{-1})$  with the continuous dual  $q^2$ -Hahn polynomials as coefficients. If  $|qz^{-1}| < 1$ , the sum is absolutely convergent and can be evaluated by means of a sum formula established in [22] (see Appendix

C):

$$(141) \quad r(\cos \theta, z) = (1 - q^2 z^{-1}) \cdot \frac{(q^2 z^{-1}, q^2 z^{-1}; q^2)_\infty}{(q z^{-1} e^{i\theta}, q z^{-1} e^{-i\theta}; q^2)_\infty}, \quad |q z^{-1}| < 1$$

where the normalisation  $\tilde{P}_0(y; q^{-2r-1}, q^{-1}, q^{-1}|q^{-2}) = q^r$  has been chosen. Within the range of validity, the rhs of (141) does not possess zeros or poles. For  $|q z^{-1}| < 1$ , the inverse transformations read:

$$(142) \quad \tilde{R}_{2, t_1+r; 0, t_1^*}^{0, t_2^*; 2, t_2+r}(z) = q^{(t_2-t_1)(2r-1)} \frac{(q^2; q^2)_{t_1+r} (q^2; q^2)_{t_2+1}}{(q^2; q^2)_{t_2+r} (q^2; q^2)_{t_1+1}} \cdot \int_0^\pi \tilde{P}_{t_1}(\cos \theta; q^{2r+1}, q, q|q^2) \tilde{P}_{t_2}(\cos \theta; q^{-2r-1}, q^{-1}, q^{-1}|q^{-2}) r(\cos \theta, z) w(\cos \theta) d\theta$$

and

$$(143) \quad \tilde{R}_{2, t_1; 0, t_1+r^*}^{0, t_2+r^*; 2, t_2}(z) = q^{(t_1-t_2)(2r+1)} \frac{(q^2; q^2)_{t_2+r+1} (q^2; q^2)_{t_1}}{(q^2; q^2)_{t_1+r+1} (q^2; q^2)_{t_2}} \cdot \int_0^\pi \tilde{P}_{t_1}(\cos \theta; q^{2r+1}, q, q|q^2) \tilde{P}_{t_2}(\cos \theta; q^{-2r-1}, q^{-1}, q^{-1}|q^{-2}) r(\cos \theta, z) w(\cos \theta) d\theta$$

with  $r \geq 0$ ,  $r(\cos \theta, z)$  given by (141) and  $w(\cos \theta) = w(\cos \theta; q^{2r+1}, 0, q, q|q^2)$  defined by (122). The last two factors in (142) and (143) may be rewritten as

$$(144) \quad r(\cos \theta, z) w(\cos \theta; q^{2r+1}, 0, q, q|q^2) = (1 - q^2 z^{-1}) (q^2 z^{-1}, q^2 z^{-1}; q^2)_\infty w(\cos \theta; q^{2r+1}, q z^{-1}, q, q|q^2)$$

According to (122), the function  $w(\cos \theta; q^{2r+1}, q z^{-1}, q, q|q^2)$  enters the orthogonality measure for the Askey-Wilson polynomials  $p_t(\cos \theta; q^{2r+1}, q z^{-1}, q, q|q^2)$ . For  $|q z^{-1}| < 1$ , the orthogonality relation reads ([20], eqs. 7.5.15-7.5.17)

$$(145) \quad \int_0^\pi p_t(\cos \theta; q^{2r+1}, q z^{-1}, q, q|q^2) p_s(\cos \theta; q^{2r+1}, q z^{-1}, q, q|q^2) w_z(\cos \theta) d\theta = \frac{\delta_{t,s}}{(q^2 z^{-1}, q^2 z^{-1}; q^2)_\infty h_t(q^{2r+1}, q z^{-1}, q, q|q^2)}$$

where  $w_z(\cos \theta) = w(\cos \theta; q^{2r+1}, q z^{-1}, q, q|q^2)$  and

$$(146) \quad h_t(q^{2r+1}, q z^{-1}, q, q|q^2) = \frac{1 - q^{2(2t+r+1)} z^{-1}}{(q^2, q^2, q^{2(r+1)}, q^{2(r+1)}, q^2 z^{-1}, q^2 z^{-1}; q^2)_t}$$

The continuous dual Hahn polynomials  $\tilde{P}_t(y; a, 0, c, d|q^2)$  and  $\tilde{P}_t(y; a^{-1}, 0, c^{-1}, d^{-1}|q^{-2})$  may be expressed in terms of the polynomials  $p_t(y; q^{2r+1}, q z^{-1}, q, q|q^2)$  making use of (129) and the connection coefficients between Askey-Wilson polynomials established in [19] (see also section 7.6 in [20]). The required relation is a special case of eqs. 7.6.2 and 7.6.3 in [20]:

$$(147) \quad \tilde{P}_t(y; a, c, d|q^2) = \sum_{s=0}^t c_{s,t} p_s(y; a, b, c, d)$$

with

$$(148) \quad c_{s,t} = b^{t-s} \frac{(cd; q^2)_t (acq^{2s}; q^2)_{t-s}}{(q^2, ad, cd, abcdq^{2(s-1)}; q^2)_s (q^2, abcdq^{4s}; q^2)_{t-s}}$$

Here the parameters  $a, b, c, d$  are only restricted by the requirement that the denominators in the polynomials and coefficients never vanish. With  $a = q^{2r+1}$ ,  $c = d = q$  and  $b = qz^{-1}$ , the relation reads

$$(149) \quad \tilde{P}_t(y; q^{2r+1}, qz^{-1}, q, q|q^2) = \sum_{s=0}^t c_{s,t}(q, z) p_s(y; q^{2r+1}, qz^{-1}, q, q|q^2)$$

with

$$(150) \quad c_{s,t}(q, z) = \left(\frac{q}{z}\right)^{t-s} \frac{(q^2; q^2)_t (q^{2(r+s+1)}; q^2)_{t-s}}{(q^{2(r+1)}, q^2, q^2, q^{2(r+s+1)}z^{-1}; q^2)_s (q^2, q^{2(r+2s+2)}z^{-1}; q^2)_{t-s}}$$

Replacing  $q$  by  $q^{-1}$  and  $z$  by  $z^{-1}$  in (149), (150) and changing from base  $q^{-2}$  to base  $q^2$  in  $p_s(y; q^{-2r-1}, q^{-1}z, q^{-1}, q^{-1}|q^{-2})$  yields

$$(151) \quad \frac{\tilde{P}_t(y; q^{-2r-1}, q^{-1}, q^{-1}|q^{-2})}{\tilde{P}_0(y; q^{-2r-1}, q^{-1}, q^{-1}|q^{-2})} = \sum_{s=0}^t c'_{s,t}(q, z) p_s(y; q^{2r+1}, qz^{-1}, q, q|q^2)$$

with

$$(152) \quad c'_{s,t}(q, z) = z^s q^{-s(3s+2r+1)} (-1)^s c_{s,t}(q^{-1}, z^{-1})$$

Here the normalisation of the polynomials  $\tilde{P}_t(y; q^{-2r-1}, q^{-1}, q^{-1}|q^{-2})$  is left arbitrary. Insertion of (145), (149) and (151) into equation (143) allows to express the R-matrix elements in terms of connection coefficients and the function specifying the corresponding orthogonality relation:

$$(153) \quad \check{R}_{2,t_1; 0, t_1+r^*}^{0, t_2+r^*; 2, t_2}(z) = q^{(t_1-t_2)(2r+1)} \frac{(q^2; q^2)_{t_2+r+1} (q^2; q^2)_{t_1}}{(q^2; q^2)_{t_1+r+1} (q^2; q^2)_{t_2}} \tilde{P}_0(y; q^{-2r-1}, q^{-1}, q^{-1}|q^{-2}) \cdot \\ (1 - q^2 z^{-1}) \sum_{t=0}^{\min(t_1, t_2)} c_{t,t_1}(q, z) c'_{t,t_2}(q, z) \frac{1}{h_t(q^{2r+1}, qz^{-1}, q, q|q^2)}$$

Equation (153) is valid for all values of  $z$  provided that the R-matrix element on the lhs is well defined. The orthogonality relations (145) take a different form for  $|qz^{-1}| > 1$  (see chapter 6 of [20]). Therefore the expression (141) does not give rise to inverse transformations of the form (142) or (143) if the condition  $|qz^{-1}| < 1$  is not satisfied.

Within the region of validity, the expression (141) for the action of  $R_{VV^*}(z)$  has a well-defined limit as  $q$  tends to zero. The latter does not show a structure which could distinguish the different irreducible components in the limit  $q \rightarrow 0$ . Hence in this region the action does not indicate an inconsistency in the quasi-particle description of the space of infinite configurations in  $\Omega_A^{(1)}$  in terms of the two type II vertex operators  $\Phi_{V_I}^{\check{V} V_I}(z)$  and  $\Phi_{V_J}^{\check{V}^* V_J}(w)$ .

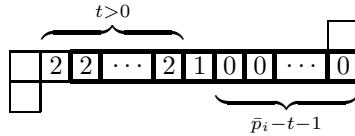
In contrast, the action of  $\check{R}_{V^*V}(z)$  on the irreducible components (118) is well-defined for all  $z \neq q^{2s}$ ,  $s \in \mathbb{Z}$ . The eigenvalue is finite at  $z = q^{2s-1} e^{\pm i\theta}$  for all  $s \in \mathbb{Z}$ . Details will be given in a forthcoming publication.

## APPENDIX A. THE MAPPING FOR VERTICAL BORDER STRIPES

A generator  $\phi_{j,t;r}$  with suitable  $j, t$  is assigned to the  $r$ -th box of a vertical border strip if this box carries the number 0 in the reference labelling defined in section 4. Hence, the normal forms corresponding to a border strip with parameters  $r_0$  and  $(p_1, \dots, p_{R-1}; \bar{p}_1, \dots, p_R)$  or  $(\emptyset; \bar{p}_1)$  can be written

$$(154) \quad \begin{aligned} & \phi_{j_{\bar{p}_1}, t_{\bar{p}_1}; s_1 + \bar{p}_1 - 1} \cdots \phi_{j_2, t_2; s_1 + 1} \phi_{j_1, t_1; s_1} \\ & \cdot \phi_{j_{\bar{p}_1 + \bar{p}_2}, t_{\bar{p}_1 + \bar{p}_2}; s_2 + \bar{p}_2 - 1} \cdots \phi_{j_{\bar{p}_1 + 2}, t_{\bar{p}_1 + 2}; s_2 + 1} \phi_{j_{\bar{p}_1 + 1}, t_{\bar{p}_1 + 1}; s_2} \\ & \cdot \phi_{j_{\bar{p}_1 + \bar{p}_2 + \bar{p}_3}, t_{\bar{p}_1 + \bar{p}_2 + \bar{p}_3}; s_3 + \bar{p}_3 - 1} \cdots \phi_{j_{\bar{p}_1 + \bar{p}_2 + 2}, t_{\bar{p}_1 + \bar{p}_2 + 2}; s_3 + 1} \phi_{j_{\bar{p}_1 + \bar{p}_2 + 1}, t_{\bar{p}_1 + \bar{p}_2 + 1}; s_3} \\ & \cdots \\ & \cdot \phi_{j_{\bar{N}}, t_{\bar{N}}; s_R + \bar{p}_R - 1} \cdots \phi_{j_{\bar{p}_1 + \dots + \bar{p}_{R-1} + 2}, t_{\bar{p}_1 + \dots + \bar{p}_{R-1} + 2}; s_R + 1} \phi_{j_{\bar{p}_1 + \dots + \bar{p}_{R-1} + 1}, t_{\bar{p}_1 + \dots + \bar{p}_{R-1} + 1}; s_R} \end{aligned}$$

where  $\bar{N} = \sum_{i=1}^R \bar{p}_i$  and the numbers  $s_i$  are defined in (49). Prescriptions for the indices  $j, t$  are obtained from modifications of figures 3-5. Each section of a border strip drawn in figure 3 is transposed such that the uppermost box of the column becomes the leftmost box of a row. In figures 4 and 5, each section is transposed such that the leftmost box of the row becomes the uppermost box of a column. Throughout the figures, the  $p_i$  is replaced by  $\bar{p}_i$  for all  $i = 1, 2, \dots, R$  and  $\bar{p}_i$  by  $p_i$  for  $i = 1, 2, \dots, R - 1$ . In the right part of the figures, each generator  $\phi_{j,t;m}^*$  is replaced by  $\phi_{j,t;m}$ . Then in figure 3 the order of the pairs  $j_i, t_i$  in the product is reversed. The modes  $m_i$  are kept unchanged. For example, the lowest part of figure 3 is changed to



$$\longleftrightarrow \begin{cases} \bar{p}_i = t + 1 : & \phi_{0,0;s_i + \bar{p}_i - 1} \cdots \phi_{0,0;s_i + 2} \phi_{0,0;s_i + 1} \\ \bar{p}_i > t + 1 : & \phi_{0,0;s_i + \bar{p}_i - 1} \cdots \phi_{0,0;s_i + \bar{p}_i - t + 1} \phi_{0,0;s_i + \bar{p}_i - t} \\ & \cdot \phi_{2,0;s_i + \bar{p}_i - t - 1} \cdots \phi_{2,0;s_i + 2} \phi_{2,0;s_i + 1} \end{cases}$$

The above changes transform figures 3, 4 and 5 to figures 3', 4' and 5', respectively. Then the factors in the  $i$ -th line in (63) are related to the numbers given to the boxes in the  $i$ -th row and the left neighbouring column by the following three steps.

- (1) For  $1 \leq i \leq R$  and  $\bar{p}_i \geq 2$ , the factors

$$\begin{aligned} & \phi_{j_{\bar{p}_1 + \dots + \bar{p}_i - 1}, t_{\bar{p}_1 + \dots + \bar{p}_i - 1}; s_i + \bar{p}_i - 2} \cdots \phi_{j_{\bar{p}_1 + \dots + \bar{p}_i - 1 + 2}, t_{\bar{p}_1 + \dots + \bar{p}_i - 1 + 2}; s_i + 1} \\ & \cdot \phi_{j_{\bar{p}_1 + \dots + \bar{p}_i - 1 + 1}, t_{\bar{p}_1 + \dots + \bar{p}_i - 1 + 1}; s_i} \end{aligned}$$

are determined according to the numbers attributed to the  $\bar{p}_i$  rightmost boxes of the  $i$ -th row. Figure 3' specifies these factors for all cases admitted by the rules (47) and (48) for the tableaux of vertical strips.

- (2) To obtain the factor  $\phi_{j_{\bar{p}_1+\dots+\bar{p}_i}, t_{\bar{p}_1+\dots+\bar{p}_i}; s_i+\bar{p}_i}$  with  $i < R$  and  $\bar{p}_i \geq 1$ , the  $i$ -th row and the  $(i+1)$ -th finite column with length  $p_{i+1}+1$  is taken into account. Figure 4' gives all possible cases together with the associated factors.
- (3) Depending on the numbers given to the boxes of the left infinite column, the leftmost factor  $\phi_{j_{\bar{N}}, t_{\bar{N}}; s_R+\bar{p}_R-1}$  is determined. All cases are found in figure 5'.

For a semi-standard super tableau of the border strip  $(\emptyset, \bar{p}_1)$  with  $\bar{p}_1 > 1$ , the normal form (61) is determined by the second and third step. In the case  $(\emptyset; 1)$ , only the last step is needed. The set of all normal forms (154) may be called  $B(p_1, \dots, p_{R-1}; \bar{p}_1, \dots, \bar{p}_R; r_0)$  if  $R > 1$  and  $B(\emptyset; \bar{p}_1)$  for  $R = 1$ . Steps 1-3 describe the one-to-one correspondence claimed in Result 2.

## APPENDIX B. THE R-MATRIX $\check{R}_{VV}(z)$

The intertwiner  $\check{R}: V_z \otimes V_w \longrightarrow V_w \otimes V_z$  satisfies the conditions

$$(155) \quad \check{R}\Delta(a) = \Delta(a)\check{R} \quad \forall a \in U_q(\widehat{gl}(2|2)).$$

This set of linear equations has a solution

$$(156) \quad \check{R}_{VV}\left(\frac{z}{w}\right) = P^{gr} R_{VV}\left(\frac{z}{w}\right)$$

where  $P^{gr}$  is the graded transposition defined in section 6 and  $R_{VV}\left(\frac{z}{w}\right)$  denotes the R-matrix in the first line of (79). In the following, the nonvanishing matrix elements  $\check{R}_{j_1, t_1; j_2, t_2}^{j_3, t_3; j_4, t_4}(z)$  of  $\check{R}_{\check{V}\check{V}}(z)$  with  $t \geq 0$  and the basis of  $\check{V}$  specified by (81), (82) are collected. They are related by

$$(157) \quad \check{R}_{j_1, t_1; j_2, t_2}^{j_3, t_3; j_4, t_4}(z) = q^{\frac{1}{8}(1-(-1)^{j_3})(1-(-1)^{j_4}) - \frac{1}{8}(1-(-1)^{j_1})(1-(-1)^{j_2})} \\ \cdot q^{-(t_1 + \frac{1}{2}\delta_{j_1,1} + \frac{1}{2}\delta_{j_1,3})^2 - (t_2 + \frac{1}{2}\delta_{j_2,1} + \frac{1}{2}\delta_{j_2,3})^2 + (t_3 + \frac{1}{2}\delta_{j_3,1} + \frac{1}{2}\delta_{j_3,3})^2 + (t_4 + \frac{1}{2}\delta_{j_4,1} + \frac{1}{2}\delta_{j_4,3})^2} \\ \cdot \left( \frac{[t_1]![t_2]!}{[t_3]![t_4]!} \right)^2 \frac{[t_1+1]^{2-2\delta_{j_1,2}+\delta_{j_1,1}-\delta_{j_1,3}} [t_2+1]^{2-2\delta_{j_2,2}+\delta_{j_2,1}-\delta_{j_2,3}}}{[t_3+1]^{2-2\delta_{j_3,2}+\delta_{j_3,1}-\delta_{j_3,3}} [t_4+1]^{2-2\delta_{j_4,2}+\delta_{j_4,1}-\delta_{j_4,3}}} \check{R}_{j_3, t_3; j_4, t_4}^{j_1, t_1; j_2, t_2}(z)$$

and

$$(158) \quad \check{R}_{j_1, t_1; j_2, t_2}^{j_3, t_3; j_4, t_4}(z) = \\ z^{t_4-t_1+\delta_{j_4,1}-\delta_{j_1,1}} \frac{[t_1+1]^{\delta_{j_1,0}-\delta_{j_1,2}} [t_2+1]^{\delta_{j_2,0}-\delta_{j_2,2}}}{[t_3+1]^{\delta_{j_3,0}-\delta_{j_3,2}} [t_4+1]^{\delta_{j_4,0}-\delta_{j_4,2}}} \check{R}_{\sigma(j_2), t_2; \sigma(j_1), t_1}^{\sigma(j_4), t_4; \sigma(j_3), t_3}(z)$$

with  $\sigma(0) = 2$ ,  $\sigma(2) = 0$ ,  $\sigma(1) = 1$ ,  $\sigma(3) = 3$  and

$$(159) \quad \check{R}_{j_1, t_1; j_2, t_2}^{j_3, t_3; j_4, t_4}(z) = \\ z^{t_4-t_1-\delta_{j_4,2}+\delta_{j_1,2}} \frac{[t_1+1]^{\delta_{j_1,1}-\delta_{j_1,3}} [t_2+1]^{\delta_{j_2,1}-\delta_{j_2,3}}}{[t_3+1]^{\delta_{j_3,1}-\delta_{j_3,3}} [t_4+1]^{\delta_{j_4,1}-\delta_{j_4,3}}} \check{R}_{\tau(j_2), t_2; \tau(j_1), t_1}^{\tau(j_4), t_4; \tau(j_3), t_3}(z)$$

with  $\tau(0) = 0$ ,  $\tau(2) = 2$ ,  $\tau(1) = 3$  and  $\tau(3) = 1$ . Here the notation  $[t]! = [t][t-1] \dots [1]$  is used.

The matrix elements  $\check{R}_{j_1, t_1; j_2, t_2}^{j_3, t_3; j_4, t_4}(z)$  are nonvanishing for  $t_1 + t_2 = t_3 + t_4$ . The solution of the intertwining condition (155) is unique up to a scalar. Normalising  $\check{R}_{\check{V}\check{V}}(z)$  by

$$(160) \quad \check{R}_{2,0; 2,0}^{2,0; 2,0}(z) = 1,$$

explicit solution of the intertwining condition (155) gives

$$(161) \quad \check{R}_{2,t_1;2,t_2}^{2,0;2,t_1+t_2}(z) = z^{t_2} q^{t_1(2t_2+1)} (q^2; q^2)_{t_2} \frac{z-1}{q^2-1} \prod_{r=0}^{t_2} \frac{q^2-1}{q^{2(t_1+r)} z - 1}$$

and

$$(162) \quad \begin{aligned} \check{R}_{2,t_1;2,t_2}^{2,s;2,t_1+t_2-s}(z) &= \frac{1}{(q^2; q^2)_s} \frac{q^{2t_1} z - 1}{z - q^{2(t_1-s)}} q^{(t_1-s)(2t_2-2s+1)-t_1(2t_2+1)} \bar{R}_{2,t_1;2,t_2}^{2,0;2,t_1+t_2}(z) \\ &\cdot \sum_{r=0}^s z^{r-s} q^{(s-r)(s-r+1)} \frac{1}{(q^2; q^2)_r (q^2; q^2)_{s-r}} \frac{(q^2; q^2)_{t_2+r}}{(q^2; q^2)_{t_2-s+r}} \left( \frac{(q^2; q^2)_{t_1}}{(q^2; q^2)_{t_1-r}} \right)^2 \\ &\cdot \prod_{r_1=0}^{s-r} (z - q^{2(t_1-r-r_1)}) \prod_{r_2=0}^r \frac{1}{q^{2(t_1-r_2)} z - 1} \end{aligned}$$

for  $t_1 \geq s > 0$  and  $t_2 - s \geq 0$ . In (161), (162) and the remainder, the notation  $(a; q^2)_n = (1 - q^{2(n-1)}a) \dots (1 - q^2a)(1 - a)$  is used. The remaining elements  $\check{R}_{2,t_1;2,t_2}^{2,s;2,t_1+t_2-s}(z)$  are obtained from the relations

$$(163) \quad \check{R}_{2,t_1;2,t_2}^{2,t_3;2,t_4}(z) = z^{t_4-t_1} \check{R}_{2,t_2;2,t_1}^{2,t_4;2,t_3}(z)$$

and

$$(164) \quad \check{R}_{2,t_1;2,t_2}^{2,t_3;2,t_4}(z) = q^{-2(t_1^2+t_2^2-t_3^2-t_4^2)} \left( \frac{(q^2; q^2)_{t_1} (q^2; q^2)_{t_2}}{(q^2; q^2)_{t_3} (q^2; q^2)_{t_4}} \right)^2 \check{R}_{2,t_3;2,t_4}^{2,t_1;2,t_2}(z)$$

which are special cases of (157)-(159). All matrix elements  $\check{R}_{j,t_1;j,t_2}^{j,t_3;j,t_4}(z)$  can be expressed in terms of  $\check{R}_{2,t_1;2,t_2}^{2,t_3;2,t_4}(z)$ :

$$(165) \quad \check{R}_{3,t_1;3,t_2}^{3,s;3,t_1+t_2-s}(z) = \frac{1}{(1-z)(q^{2(t_2+1)}-1)} \cdot \left\{ q^{s-t_1} (q^{2(s-t_1)}-1) (q^{2(t_1+t_2-s+1)}-1) \check{R}_{2,t_1;2,t_2+1}^{2,s;2,t_1+t_2-s+1}(z) \right. \\ \left. + q^{t_1+2t_2-3s-1} (z - q^{2(s-t_1+1)}) (q^{2(s+1)}-1) \check{R}_{2,t_1;2,t_2+1}^{2,s+1;2,t_1+t_2-s}(z) \right\}$$

$$(166) \quad \begin{aligned} \check{R}_{0,t_1;0,t_2}^{0,t_3;0,t_4}(z) &= q^{t_3+t_4-t_1-t_2} \frac{(q^{2(t_1+1)}-1)(q^{2(t_2+1)}-1)}{(q^{2(t_3+1)}-1)(q^{2(t_4+1)}-1)} \check{R}_{2,t_1;2,t_2}^{2,t_3;2,t_4}(z) \\ \check{R}_{1,t_1;1,t_2}^{1,t_3;1,t_4}(z) &= q^{t_3+t_4-t_1-t_2} \frac{(q^{2(t_1+1)}-1)(q^{2(t_2+1)}-1)}{(q^{2(t_3+1)}-1)(q^{2(t_4+1)}-1)} \check{R}_{3,t_1;3,t_2}^{3,t_3;3,t_4}(z) \end{aligned}$$

The R-matrix elements  $\check{R}_{j_1,t_1;j_2,t_2}^{j_3,t_3;j_4,t_4}(z)$  with  $j_1 = j_3 \neq j_2 = j_4$  or  $j_1 = j_4 \neq j_2 = j_3$  are given by

$$(167) \quad \check{R}_{2,t_1;3,t_2}^{2,s;3,t_1+t_2-s}(z) = \frac{q^{t_1+1}}{q^{2(t_1+t_2+2)}-1} \cdot \left\{ q^{t_1+1} (q^{2(t_2+1)}-1) \check{R}_{2,t_1;2,t_2}^{2,s;2,t_1+t_2-s}(z) + q^{-s-1} (q^{2(s+1)}-1) \check{R}_{3,t_1;3,t_2}^{3,s;3,t_1+t_2-s}(z) \right\}$$

$$(168) \quad \check{R}_{3,t_1;2,t_2}^{3,s;2,t_1+t_2-s}(z) = \frac{q^{s+1}}{q^{2(t_1+t_2+2)} - 1} \\ \cdot \left\{ q^{-t_1-1} (q^{2(t_1+1)} - 1) \check{R}_{2,t_1;2,t_2}^{2,s_1;2,s_2}(z) + q^{s+1} (q^{2(t_1+t_2-s+1)} - 1) \check{R}_{3,t_1;3,t_2}^{3,s;3,t_1+t_2-s}(z) \right\}$$

$$(169) \quad \check{R}_{2,t_1;3,t_2}^{3,s;2,t_1+t_2-s}(z) = \frac{q^{s-t_2+t_1+1}}{q^{2(t_1+t_2+2)} - 1} \\ \cdot \left\{ (q^{2(t_2+1)} - 1) \check{R}_{2,t_1;2,t_2}^{2,s;2,t_1+t_2-s}(z) - q^{s-t_1} (q^{2(t_1+t_2-s+1)} - 1) \check{R}_{3,t_1;3,t_2}^{3,s;3,t_1+t_2-s}(z) \right\}$$

$$(170) \quad \check{R}_{3,t_1;2,t_2}^{2,s;3,t_1+t_2-s}(z) = \left( \frac{z}{q} \right)^{t_2-s} \check{R}_{2,t_2;3,t_1}^{3,t_1+t_2-s;2,s}(z)$$

$$(171) \quad \check{R}_{2,t_1;3,t_2}^{3,s;2,t_1+t_2-s}(z) = \\ \frac{q^{2(s+1)} - 1}{q^{2(t_2+1)} - 1} \check{R}_{2,t_1;1,t_2}^{1,s;2,t_1+t_2-s}(z) = q^{2(s-t_2)} \frac{q^{2(t_1+t_2-s+1)} - 1}{q^{2(t_1+1)} - 1} \check{R}_{0,t_1;3,t_2}^{3,s;0,t_1+t_2-s}(z) = \\ \frac{(q^{2(s+1)} - 1)(q^{2(t_1+t_2-s+1)} - 1)}{(q^{2(t_1+1)} - 1)(q^{2(t_2+1)} - 1)} \check{R}_{0,t_1;1,t_2}^{1,s;0,t_1+t_2-s}(z)$$

$$(172) \quad \check{R}_{3,t_1;2,t_2}^{2,s;3,t_1+t_2-s}(z) = \\ \frac{q^{2(s+1)} - 1}{q^{2(t_2+1)} - 1} \check{R}_{3,t_1;0,t_2}^{0,s;3,t_1+t_2-s}(z) = q^{2(s-t_2)} \frac{q^{2(t_1+t_2-s+1)} - 1}{q^{2(t_1+1)} - 1} \check{R}_{1,t_1;2,t_2}^{2,s;1,t_1+t_2-s}(z) = \\ \frac{(q^{2(s+1)} - 1)(q^{2(t_1+t_2-s+1)} - 1)}{(q^{2(t_1+1)} - 1)(q^{2(t_2+1)} - 1)} \check{R}_{1,t_1;0,t_2}^{0,s;1,t_1+t_2-s}(z)$$

$$(173) \quad \check{R}_{2,t_1;3,t_2}^{2,s;3,t_1+t_2-s}(z) = \frac{(q^{2(s+1)} - 1)(q^{2(t_1+t_2-s+1)} - 1)}{q^{2(t_1+1)} - 1} \check{R}_{0,t_1;1,t_2}^{0,s;1,t_1+t_2-s}(z) = \\ q^{s-t_1} z^{t_2-s+1} \frac{q^{2(t_1+t_2-s+1)} - 1}{q^{2(t_2+1)} - 1} \check{R}_{1,t_2;2,t_1}^{1,t_1+t_2-s;2,s}(z) = \\ q^{t_1-s} z^{t_2-s} \frac{q^{2(s+1)} - 1}{q^{2(t_1+1)} - 1} \check{R}_{3,t_2;0,t_1}^{3,t_1+t_2-s;0,s}(z)$$

$$(174) \quad \check{R}_{3,t_1;2,t_2}^{3,s;2,t_1+t_2-s}(z) = \frac{(q^{2(s+1)} - 1)(q^{2(t_1+t_2-s+1)} - 1)}{q^{2(t_1+1)} - 1} \check{R}_{1,t_1;0,t_2}^{1,s;0,t_1+t_2-s}(z) = \\ q^{t_1-s} z^{t_2-s-1} \frac{q^{2(s+1)} - 1}{q^{2(t_1+1)} - 1} \check{R}_{2,t_2;1,t_1}^{2,t_1+t_2-s;1,s}(z) = \\ q^{s-t_1} z^{t_2-s} \frac{q^{2(t_1+t_2-s+1)} - 1}{q^{2(t_2+1)} - 1} \check{R}_{0,t_2;3,t_1}^{0,t_1+t_2-s;3,s}(z)$$



$$(175) \quad \check{R}_{2,t_2;0,t_1}^{0,t_1+t_2-s;2,s}(z) = z^{s-t_2} \check{R}_{0,t_1;2,t_2}^{2,s;0,t_1+t_2-s}(z) =$$

$$\frac{q^{t_2} z^{s-t_2}}{q^{2(t_1+t_2+1)} - 1} \left\{ q^{t_1-s+1} \frac{(q^{2(t_1+1)} - 1)(q^{2t_2} - 1)}{q^{2(t_1+t_2-s+1)} - 1} \check{R}_{2,t_1;3,t_2-1}^{3,s-1;2,t_1+t_2-s}(z) \right.$$

$$\left. + (q^{2(t_1+1)} - 1) \check{R}_{3,t_1;2,t_2}^{2,s;3,t_1+t_2-s}(z) \right\}$$

$$(176) \quad \check{R}_{3,t_2;1,t_1}^{1,t_1+t_2-s;3,s}(z) = z^{s-t_2} \check{R}_{1,t_1;3,t_2}^{3,s;1,t_1+t_2-s}(z) =$$

$$- \frac{q^{t_2+1} z^{s-t_2}}{q^{2(t_1+t_2+2)} - 1} \left\{ q^{t_1-s} \frac{(q^{2(t_1+1)} - 1)(q^{2(t_2+1)} - 1)}{q^{2(t_1+t_2-s+1)} - 1} \check{R}_{3,t_1;2,t_2}^{2,s;3,t_1+t_2-s}(z) \right.$$

$$\left. + (q^{2(t_1+1)} - 1) \check{R}_{2,t_1+1;3,t_2}^{3,s;2,t_1+t_2-s+1}(z) \right\}$$

$$(177) \quad \check{R}_{2,t_2;0,t_1}^{2,t_1+t_2-s;0,s}(z) = z^{s-t_2+1} \check{R}_{0,t_1;2,t_2}^{0,s;2,t_1+t_2-s}(z)$$

$$= z^{s-t_2+1} \frac{q^{2(t_1+1)} - 1}{q^{2(t_1+t_2+1)} - 1} \left\{ q^{t_1+2t_2-s} \check{R}_{3,t_1;2,t_2}^{3,s;2,t_1+t_2-s} \right.$$

$$\left. + \frac{q^{2t_2} - 1}{q^{2(s+1)} - 1} \check{R}_{2,t_1;3,t_2-1}^{2,s;3,t_1+t_2-s-1}(z) \right\}$$

$$(178) \quad \check{R}_{3,t_2;1,t_1}^{3,t_1+t_2-s;1,s}(z) = z^{s-t_2} q^{2(s-t_1)} \frac{(q^{2(t_1+1)} - 1)(q^{2(t_1+t_2-s+1)} - 1)}{(q^{2(s+1)} - 1)(q^{2(t_2+1)} - 1)} \check{R}_{3,t_1;1,t_2}^{3,s;1,t_1+t_2-s}(z)$$

$$(179) \quad \check{R}_{3,t_1;1,t_2}^{3,s;1,t_1+t_2-s}(z) = \frac{q^{2(t_2+1)} - 1}{q^{2(t_1+t_2+2)} - 1}$$

$$\cdot \left\{ \check{R}_{3,t_1;2,t_2+1}^{3,s;2,t_1+t_2-s+1}(z) + q^{t_1+2t_2-s+2} \frac{q^{2(t_1+1)} - 1}{q^{2(t_1+t_2-s+1)} - 1} \check{R}_{2,t_1;3,t_2}^{2,s;3,t_1+t_2-s}(z) \right\}$$

$$(180) \quad \check{R}_{1,t_2;3,t_1}^{1,t_1+t_2-s;3,s}(z) = z^{s-t_2} q^{2(t_1-s)} \frac{(q^{2(s+1)} - 1)(q^{2(t_2+1)} - 1)}{(q^{2(t_1+1)} - 1)(q^{2(t_1+t_2-s+1)} - 1)} \check{R}_{1,t_1;3,t_2}^{1,s;3,t_1+t_2-s}(z)$$

$$(181) \quad \check{R}_{1,t_1;3,t_2}^{1,s;3,t_1+t_2-s}(z) =$$

$$\frac{1}{z} \frac{q^{-t_1+2t_2+s+1}}{q^{2(t_1+t_2+2)} - 1} \frac{q^{2(t_1+1)} - 1}{q^{2(s+1)} - 1} \cdot \left\{ (q^{2(t_1+1)} - 1) \check{R}_{2,t_1;3,t_2}^{2,s;3,t_1+t_2-s}(z) \right.$$

$$\left. + q^{-t_1-2t_2+s-1} (q^{2(t_1+t_2-s+1)} - 1) \check{R}_{3,t_1;2,t_2+1}^{3,s;2,t_1+t_2-s+1}(z) \right\}$$

The above expressions are valid for  $0 \leq s \leq t_1+t_2$ . In (177), the second contribution on the rhs is dropped for  $s = t_1+t_2$ . The remaining nonvanishing R-matrix elements

are

$$\begin{aligned}
 (182) \quad \check{R}_{3,t_2;1,t_1}^{0,t_1+t_2-s+1;2,s}(z) = & \\
 q^{2(t_2-s)+1} \frac{(q^{2(t_1+1)} - 1)(q^{2(s+1)} - 1)}{(q^{2(t_2+1)} - 1)(q^{2(t_1+t_2-s+2)} - 1)} \check{R}_{1,t_2;3,t_1}^{2,t_1+t_2-s+1;0,s}(z) = & \\
 z^{s-t_2-1} q^{t_2-t_1} \frac{q^{2(t_1+1)} - 1}{q^{2(t_2+1)} - 1} \check{R}_{3,t_1;1,t_2}^{2,s;0,t_1+t_2-s+1}(z) = & \\
 z^{s-t_2} q^{t_1+t_2-2s+1} \frac{q^{2(s+1)} - 1}{q^{2(t_1+t_2-s+2)} - 1} \check{R}_{1,t_1;3,t_2}^{0,s;2,t_1+t_2-s+1}(z) &
 \end{aligned}$$

$$\begin{aligned}
 (183) \quad \check{R}_{1,t_1;3,t_2}^{0,s;2,t_1+t_2-s+1}(z) = & \frac{q^{2(t_1+1)} - 1}{q^{2(t_1+t_2+2)} - 1} \\
 \cdot \left\{ q^{t_1+2t_2-s+2} \check{R}_{2,t_1+1;3,t_2}^{3,s;2,t_1+t_2-s+1}(z) - \frac{q^{2(t_2+1)} - 1}{q^{2(s+1)} - 1} \check{R}_{3,t_1;2,t_2}^{2,s;3,t_1+t_2-s}(z) \right\} &
 \end{aligned}$$

with  $0 \leq s \leq t_1 + t_2 + 1$  and

$$\begin{aligned}
 (184) \quad \check{R}_{2,t_2;0,t_1}^{1,t_1+t_2-s;3,s-1}(z) = & \\
 q^{2(t_2-s)+1} \frac{(q^{2(t_1+1)} - 1)(q^{2s} - 1)}{(q^{2(t_2+1)} - 1)(q^{2(t_1+t_2-s+1)} - 1)} \check{R}_{0,t_2;2,t_1}^{3,t_1+t_2-s;1,s-1}(z) = & \\
 z^{s-t_2} q^{t_1+t_2-2s+1} \frac{q^{2s} - 1}{q^{2(t_1+t_2-s+1)} - 1} \check{R}_{0,t_1;2,t_2}^{1,s-1;3,t_1+t_2-s}(z) = & \\
 z^{s-t_2-1} q^{t_2-t_1} \frac{q^{2(t_1+1)} - 1}{q^{2(t_2+1)} - 1} \check{R}_{2,t_1;0,t_2}^{3,s-1;1,t_1+t_2-s}(z) &
 \end{aligned}$$

$$\begin{aligned}
 (185) \quad \check{R}_{2,t_1;0,t_2}^{3,s-1;1,t_1+t_2-s}(z) = & \frac{q^{2(t_2+1)} - 1}{q^{2(t_1+t_2+1)} - 1} \\
 \cdot \left\{ q^{t_1+2t_2-s+2} \frac{q^{2t_1} - 1}{q^{2(t_1+t_2-s+1)} - 1} \check{R}_{3,t_1-1;2,t_2}^{2,s-1;3,t_1+t_2-s}(z) - \check{R}_{2,t_1;3,t_2}^{3,s-1;2,t_1+t_2-s+1}(z) \right\} &
 \end{aligned}$$

with  $t_1 + t_2 > 0$  and  $1 \leq s \leq t_1 + t_2$ . In the limit  $q \rightarrow 0$ , well-defined expressions are found for the R-matrix elements with respect to a different basis  $\{\tilde{v}_{j,t}\}_{0 \leq j \leq 3, t \in \mathbb{N}_0}$  given by

$$(186) \quad \tilde{v}_{j,t} = q^{t(\alpha_j)} v_{j,t}$$

with  $\alpha_j = \frac{1}{2} + \delta_{j,1} - \delta_{j,2}$ . The matrix elements  $\tilde{R}_{j_1,t_1;j_2,t_2}^{j_3,t_3;j_4,t_4}(z)$  of the corresponding R-matrix are related to the above matrix elements by

$$(187) \quad \tilde{R}_{j_1,t_1;j_2,t_2}^{j_3,t_3;j_4,t_4}(z) = q^{t_1(t_1+\alpha_{j_1})+t_2(t_2+\alpha_{j_2})-t_3(t_3+\alpha_{j_3})-t_4(t_4+\alpha_{j_4})} \bar{R}_{j_1,t_1;j_2,t_2}^{j_3,t_3;j_4,t_4}(z)$$

Straightforward analysis of (161)-(185) with properties (157)-(159) yields

$$(188) \quad \lim_{q \rightarrow 0} \tilde{R}_{j_1,t_1;j_2,t_2}^{j_3,t_3;j_4,t_4}(z) = \delta_{j_1,j_4} \delta_{j_2,j_3} \delta_{t_1,t_4} \delta_{t_2,t_3} z^{t_2+\theta_{j_2,j_1}}$$

APPENDIX C. EVALUATION OF  $r(y, z)$ 

The function  $r(y, z)$  in (139) and (140) can be obtained by collecting all contributions to the term  $v_{0,0}^* \otimes v_{2,r}$  on the rhs of (140):

$$(189) \quad r(y, z) = \frac{1}{\tilde{P}_0(y; q^{-2r-1}, q^{-1}, q^{-1}|q^{-2})} \cdot \sum_{t=0}^{\infty} q^{t(2r+1)} \frac{(q^2; q^2)_{t+1} (q^2; q^2)_r}{(q^2; q^2)_{t+r} (q^2; q^2)_1} \tilde{P}_t(y; q^{2r+1}, q, q|q^2) \check{R}_{2,t+r;0,t^*}^{0,0^*;2,r}(z)$$

A choice of  $\tilde{P}_0(y; q^{-2r-1}, q^{-1}, q^{-1}|q^{-2})$  sets the normalisation of the polynomials  $\tilde{P}_t(y; q^{-2r-1}, q^{-1}, q^{-1}|q^{-2})$ . Its value may depend on  $r$ . Comparison of the intertwining conditions yields a relation between the matrix elements  $\check{R}_{2,t_1;0,t_2^*}^{0,t_3^*;2,t_1-t_2+t_3}(z)$  or  $\check{R}_{0,t_2^*;2,t_1}^{2,t_1-t_2+t_3;0,t_3^*}(z)$  and matrix elements of  $\check{R}_{VV}(q^2 z^{-1})$ :

$$(190) \quad \begin{aligned} \check{R}_{2,t_1;0,t_2^*}^{0,t_3^*;2,t_1-t_2+t_3}(z) &= q^{t_3-t_2} \frac{[t_3+1]}{[t_2+1]} \check{R}_{2,t_3;2,t_1}^{2,t_1-t_2+t_3;2,t_2}(q^2 z^{-1}) \\ \check{R}_{0,t_2^*;2,t_1}^{2,t_1-t_2+t_3;0,t_3^*}(z) &= q^{t_2-t_3} \frac{[t_3+1]}{[t_2+1]} \check{R}_{2,t_1;2,t_3}^{2,t_2;2,t_1-t_2+t_3}(q^2 z^{-1}) \end{aligned}$$

Here an overall normalisation is fixed by equating the R-matrix elements on the lhs with those on rhs for  $t_1 = t_2 = t_3 = 0$ . Using (190) with (157), (161), the sum in (189) can be written

$$(191) \quad q^r \frac{1 - q^2 z^{-1}}{1 - q^{2(r+1)} z^{-1}} \cdot \sum_{t=0}^{\infty} \tilde{P}_t(y; q^{2r+1}, q, q|q^2) \frac{(q^{2(r+1)}; q^2)_t}{(q^{2(r+2)} z^{-1}; q^2)_t} (q z^{-1})^t$$

The infinite sum in (191) is related to a generating function for Askey-Wilson polynomials derived in [22]. In base  $q^2$ , eq. 2.10 in [22] reads

$$(192) \quad \sum_{t=0}^{\infty} \frac{(abcd; q^2)_{2t} p_t(\cos \theta; a, b, c, d|q^2)}{(q^2, ab, ac, bc, abcdq^{2(t-1)}; q^2)_t} \frac{(f/g, abc/f; q^2)_t}{(abcdg/f, fd; q^2)_t} g^t = \frac{(abcd, dg, abcdge^{i\theta}/f, fe^{i\theta}; q^2)_{\infty}}{(abcdg/f, df, abce^{i\theta}, ge^{i\theta}; q^2)_{\infty}} {}_8W_7(abce^{i\theta}/q^2; ae^{i\theta}, be^{i\theta}, ce^{i\theta}, f/g, abc/f; q^2, ge^{-i\theta})$$

for  $|g| < 1$ . For the definitions of the very-well-poised basic hypergeometric series  ${}_8W_7(a_1; a_4, a_5, a_6, a_7, a_8; q^2, \alpha)$ , the reader is referred to [20]. The orthonormal continuous dual  $q^2$ -Hahn polynomials  $\tilde{P}_t(\cos \theta; a, c, d|q^2)$  are related to the Askey-Wilson polynomials by

$$(193) \quad \tilde{P}_t(\cos \theta; a, c, d|q^2) = \frac{p_t(\cos \theta; a, 0, c, d|q^2)}{(q^2, ac; q^2)_t}$$

For  $b = 0$ , equation (192) simplifies:

$$(194) \quad \sum_{t=0}^{\infty} \tilde{P}_t(\cos \theta; a, c, d|q^2) \frac{(f/g; q^2)_t}{(df; q^2)_t} g^t = \frac{(dg, fe^{i\theta}; q^2)_{\infty}}{(df, ge^{i\theta}; q^2)_{\infty}} {}_3\phi_2(ae^{i\theta}, ce^{i\theta}, f/g; ac, fe^{i\theta}; q^2, ge^{-i\theta}), \quad |g| < 1$$

With  $a = q^{2r+1}$ ,  $c = d = q$ ,  $f = q^{2r+3}z^{-1}$  and  $g = qz^{-1}$ , the rhs of (194) coincides with the sum (191) while the rhs becomes

$$(195) \quad \frac{(qg, fe^{i\theta}; q^2)_\infty}{(qf, ge^{i\theta}; q^2)_\infty} {}_2\phi_1(q^{2r+1}e^{i\theta}, qe^{i\theta}; q^{2r+3}z^{-1}e^{i\theta}; q^2, qz^{-1}e^{-i\theta})$$

The basic hypergeometric series in (195) has the form  ${}_2\phi_1(A, B; C; q^2, C/AB)$ . If  $|C/AB| < 1$ , the latter can be summed according to eq. 1.5.1 in [20]:

$$(196) \quad {}_2\phi_1(A, B; C; q^2, C/AB) = \frac{(C/A, C/B; q^2)_\infty}{(C, C/AB; q^2)_\infty}$$

Use of (196) in (195) and collecting terms yields

$$(197) \quad r(y, z) = \frac{1}{\tilde{P}_0(y; q^{-2r-1}, q^{-1}, q^{-1}|q^{-2})} q^r (1 - q^2 z^{-1}) \frac{(q^2 z^{-1}, q^2 z^{-1}; q^2)_\infty}{(qz^{-1}e^{i\theta}, qz^{-1}e^{-i\theta}; q^2)_\infty},$$

$|qz^{-1}| < 1$

Setting  $\tilde{P}_0(y; q^{-2r-1}, q^{-1}, q^{-1}|q^{-2}) = q^r$  leads to (141). Equations (194) and (196) allow to handle the contributions to the terms  $v_{0,s}^* \otimes v_{2,r+s}$  on the rhs of (140) or  $v_{0,r+s}^* \otimes v_{2,s}$  on the rhs of (139) in a similar way. The condition  $qz^{-1} < 1$  proves sufficient to reproduce the result (141) in all cases.

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